LECTURE 10 — GRA 6035

EIVIND ERIKSEN — NOV 1ST 2012 — MATHEMATICS

PLAN:

1. Example: Kuhn-Tucker problem
2. Differential equations
   - Introduction
   - First order seperable
   - (- First order linear) ← Next week

Reading:

[FMEA] 5.1 - 5.4

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1. $$\boxed{\min \underbrace{2x^2+y^2+3z^2}_{f(x,y,z)} \text{ subj. to } \begin{array}{l} x-y+2z \geq 3 \\ x+y \geq 3 \end{array}}$$ KT probl. not in std. form (max + ≤)

$$= \boxed{\max \underbrace{-2x^2-y^2-3z^2}_{-f(x,y,z)} \text{ subj. to } \begin{array}{l} -x+y-2z \leq -3 \\ -x-y \leq -3 \end{array}}$$ KT probl. in std. form

$$\mathcal{L} = -2x^2-y^2-3z^2-\lambda_1\cdot(-x+y-2z)-\lambda_2(-x-y)$$

$$\begin{array}{l} \mathcal{L}'_x = \\ \mathcal{L}'_y = \\ \mathcal{L}'_z = \end{array} \underbrace{\boxed{\begin{array}{l} -4x+\lambda_1+\lambda_2=0 \\ -2y-\lambda_1+\lambda_2=0 \\ -6z+2\lambda_1 = 0 \end{array}}}_{\text{FOC}}$$

$$\boxed{\begin{array}{l} x-y+2z \geq 3 \\ x+y \geq 3 \end{array}} \quad C$$

$$\boxed{\begin{array}{l} \lambda_1 \geq 0,\ \lambda_2 \geq 0 \\ \text{if } x-y+2z>3 \text{ then } \lambda_1=0 \\ \text{if } x+y>3 \text{ then } \lambda_2=0 \end{array}} \quad CSC$$

KT conditions = FOC + C + CSC

Solve KT conditions:

case a)	b)	c)	d)
$x-y+2z=3$	$x-y+2z=3$	$x-y+2z>3$	$x-y+2z>3$
$x+y=3$	$x+y>3$	$x+y=3$	$x+y>3$
$\lambda_1\geq 0$	$\lambda_1\geq 0$	$\lambda_1=0$	$\lambda_1=0$
$\lambda_2\geq 0$	$\lambda_2=0$	$\lambda_2\geq 0$	$\lambda_2=0$
$-4x+\lambda_1+\lambda_2=0$	same as $\leftarrow$	same as $\leftarrow$	same as $\leftarrow$
$-2y-\lambda_1+\lambda_2=0$			
$-6z+2\lambda_1=0$			

Case a) $\quad x=\dfrac{\lambda_1+\lambda_2}{4} \quad y=\dfrac{-\lambda_1+\lambda_2}{2} \quad z=\dfrac{\lambda_1}{3}$

$$\frac{\lambda_1+\lambda_2}{4}-\frac{-\lambda_1+\lambda_2}{2}+2\,\frac{\lambda_1}{3}=3 \quad |\cdot 12$$

$$3(\lambda_1+\lambda_2)-6(-\lambda_1+\lambda_2)+8\lambda_1=36$$

$$\boxed{17\lambda_1-3\lambda_2=36}$$

$$\frac{\lambda_1+\lambda_2}{4}+\frac{-\lambda_1+\lambda_2}{2}=3 \quad |\cdot 4$$

$$\lambda_1+\lambda_2+2(-\lambda_1+\lambda_2)=12$$

$$\boxed{-\lambda_1+3\lambda_2=12}$$

Add eqn's: $16\lambda_1=48 \Rightarrow \underline{\lambda_1=3}$

$-3+3\lambda_2=12 \Rightarrow \underline{\lambda_2=5}$

both $\lambda_1,\lambda_2\geq 0$ ok.

$\boxed{x=2 \quad y=1 \quad z=1 \quad \lambda_1=3 \quad \lambda_2=5}$ $\quad f=12$

Case b) $x - y + 2z = 3$ $\qquad \lambda_1/4 + \lambda_1/2 + 2\,\lambda_1/3 = 3 \quad |\cdot 12$

$\lambda_2 = 0$ $\qquad \Uparrow$

$3\lambda_1 + 6\lambda_1 + 8\lambda_1 = 36$

FOC: $-4x + \lambda_1 = 0 \qquad x = \lambda_1/4$

$-2y - \lambda_1 = 0 \qquad y = -\lambda_1/2$ $\qquad 17\lambda_1 = 36$

$-6z + 2\lambda_1 = 0 \qquad z = \lambda_1/3$ $\qquad \lambda_1 = 36/17 \geq 0$ ok.

$\lambda_1 \geq 0$

$x + y > 3$: $\underbrace{\frac{36/17}{4}}_{x} + \left(-\underbrace{\frac{36/17}{2}}_{y}\right) = \frac{9}{17} - \frac{18}{17} = -\frac{9}{17} \not> 3$

no sol's in case b)

Case c) $\lambda_1 = 0$

$x + y = 3 \longrightarrow \lambda_2/4 + \lambda_2/2 = 3 \quad |\cdot 4$

FOC: $-4x + \lambda_2 = 0 \qquad x = \lambda_2/4 \qquad \lambda_2 + 2\lambda_2 = 12$

$-2y + \lambda_2 = 0 \qquad y = \lambda_2/2 \qquad 3\lambda_2 = 12 \quad \lambda_2 = 4$

$-6z = 0 \qquad z = 0 \qquad x = 1,\ y = 2,\ z = 0$

$\lambda_2 \geq 0$ ok

$x - y + 2z > 3 \qquad 1 - 2 + 2\cdot 0 = -1 \not> 3$

no sol's in case c)

Case d): $\left.\begin{array}{l}\lambda_1 = 0\\ \lambda_2 = 0\end{array}\right\}$ $x = y = z = 0$

$x - y + 2z > 3 \qquad 0 \not> 3$ $\qquad$ no sol's in case d)

$x + y \qquad > 3$

Conclusion: one solution to KT conditions

$x = 2,\ y = 1,\ z = 1;\ \lambda_1 = 3,\ \lambda_2 = 5$ $\qquad f = 12$

Is this min?

Candidate: $(x^*, y^*, z^*; \lambda_1^*, \lambda_2^*) = (2,1,1;3,5)$

$\mathcal{L}(x,y,z;3,5) = -2x^2-y^2-3z^2-3(-x+y-2z)-5(-x-y)$

(use $\mathcal{L}$ from the KT pbl. in std. form, want to show that this function is concave)

$$\mathcal{L}'' = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

negative definite
$(D_1 = -4, D_2 = 8, D_3 = -48)$

$\Downarrow$

$\mathcal{L}(x,y,z;3,5)$ is concave

$\Downarrow$

$(2,1,1)$ is max for $-f$

$\Updownarrow$

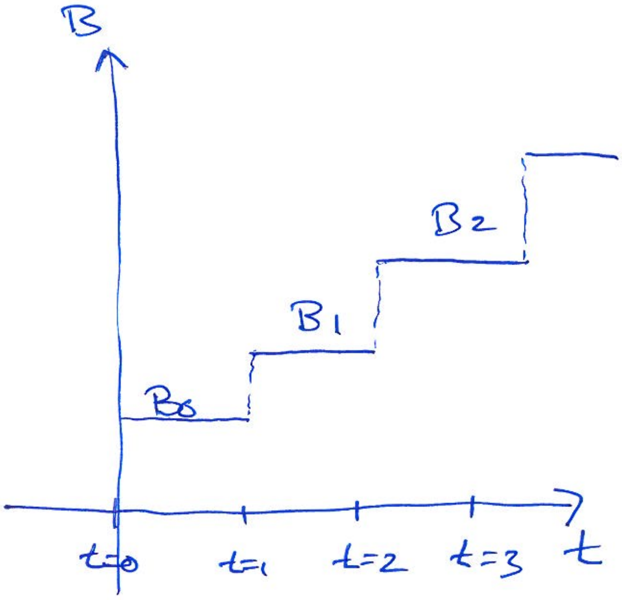
$(2,1,1)$ is min for f (min value is $f=12$)

(2) Differential equations

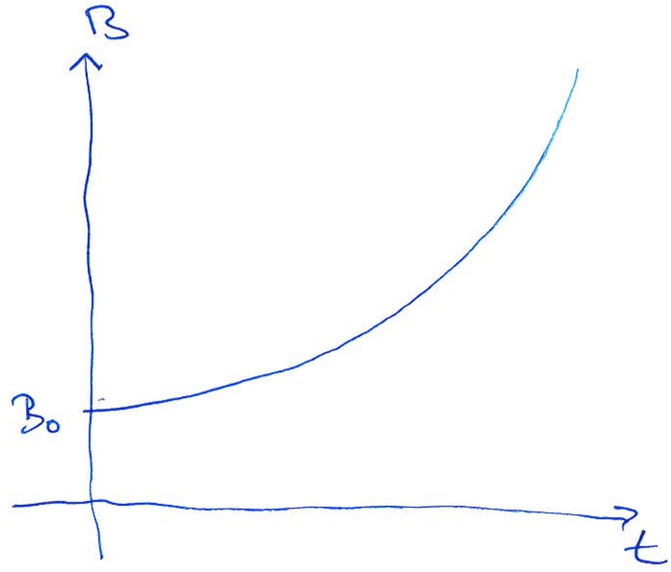
Ex: Bank account

$B(t)$: balance at time t

Dynamics = changes in $B(t)$ = interest/yield



(discrete time)



(continuous time)

$$B_{t+1} = (1+r)B_t = B_t + r \cdot B_t$$

$$\underbrace{\boxed{B_{t+1} - B_t = r B_t}}_{\text{change}}$$

(difference equation)

$$\frac{B(t+h) - B(t)}{h} \simeq B'(t)$$

(h small)

$$\boxed{B'(t) = r B(t)}$$

(differential equation)

$B'(t)$ = change

A differential equation is an equation that relates an unknown function with its derivatives.

- We only consider functions in one variable (ordinary differential equations).
- Notation: $y = y(t)$ $\quad y' = \dot{y}, \; y'' = \ddot{y}, \; \ldots$

* First order: Only involves y and y' (and t)
* Second order: Involves y, y', y'' (and t)

Ex:

$$\left.\begin{array}{l} y' = t^2 - 3 \\ y' + y = t^2 - 3 \end{array}\right\} \text{first order}$$

$$\left. y'' + y' + y = t \right\} \text{second order}$$

$$\left. (y')^2 - 2y\dot{y} = e^y \cdot t \right\} \text{first order, difficult}$$

1) First order diff-eqn's solvable by direct integration

Ex: $y' = t^2 - 3 \quad \Big| \int \_\, dt$

$$\int y'\,dt = \int (t^2 - 3)\,dt$$

See separate note to review integration $\longrightarrow$ $\underline{y = \frac{1}{3}t^3 - 3t + C}$

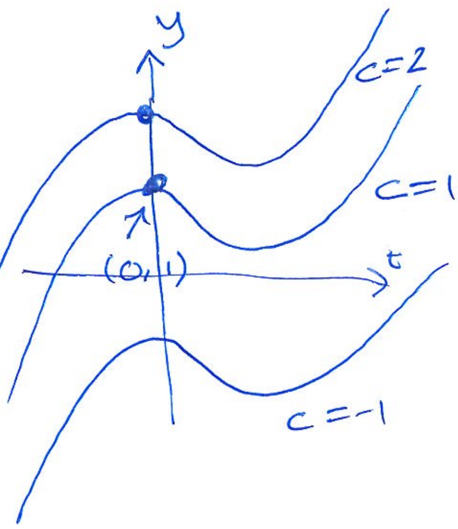
Check:

LHS: $y' = t^2 - 3$
RHS: $t^2 - 3$ $\Big\}$ ok.

In general: A diff. eqn. of first order can be solved by direct integration if it can be written in the form

$$\boxed{y' = f(t)} \Longrightarrow y = \int f(t)\,dt$$

Fact: The general solution of a first order differential eqn. will contain one undetermined constant C.



$$y' = t^2 - 3$$
$$y = \frac{1}{3}t^3 - 3t + C$$

y is determined by

$\boxed{\text{differential eqn.}}$ + $\boxed{\text{initial condition}}$

change — starting point

Ex: $y' = t^2 - 3$, $y(0) = 1$
($t=0, y=1$)

$$y = \int t^2 - 3 \, dt$$
$$y = \frac{1}{3}t^3 - 3t + C$$
(general solution)

$y(0) = 1$: $t = 0, y = 1$

$$1 = \frac{1}{3}\cdot 0^3 - \cancel{3\cdot 0} + C$$
$$C = 1$$

Particular solution: $y = \frac{1}{3}t^3 - 3t + 1$

Fact: The general solution of a second order diff. eqn. will contain two undetermined constants; C_1 and C_2.

2) First order seperable differential equations

Ex: $y' = ry \quad | \cdot \frac{1}{y}$

$\frac{1}{y} \cdot y' = r \quad | \int \ldots dt$

$\int \frac{1}{y} \underbrace{y' \, dt} = \int r \, dt$

$\int \frac{1}{y} \, dy = rt + C$

$\boxed{\ln|y| = rt + C}$

(implicit form of the solution)

Substitution:

$u = y(t)$

$du = u' \cdot dt$

$du = y'(t) \, dt \Rightarrow dt = \frac{du}{y'}$

$\int \frac{1}{y} y' \, dt = \int \frac{1}{u} \frac{du}{\cancel{y'}} \cancel{y'}$

$= \int \frac{1}{u} \, du$

$\boxed{y' \, dt = dy}$

$\exp(\ln|y|) = \exp(rt + C)$

$e^{\ln|y|} = e^{rt+C}$

$|y| = e^{rt} \cdot e^{C}$

$y = \pm e^{rt} \cdot e^{C}$

$\boxed{y = K \cdot e^{rt}} \quad (K = \pm e^{C})$

(explicit solution)

In general: A first order diff. eqn. is seperable if it can be written in the form

$$\boxed{y' = f(y) \cdot g(t)}$$

$\frac{1}{f(y)} \cdot y' = g(t)$

$\int \frac{1}{f(y)} y' \, dt = \int g(t) \, dt \Rightarrow \boxed{\int \frac{1}{f(y)} \, dy = \int g(t) \, dt}$

In general, when we have solved the two integrals in

$$\int \frac{1}{f(y)}\,dy = \int g(t)\,dt$$

we have an <u>implicit form</u> of the solution. To find the general solution explicitly, we have to solve for y.

<u>Ex:</u>

$$y' = 3y^2 - 2ty^2$$

$$y' = y^2 \cdot (3-2t)$$

$$\frac{1}{y^2}\, y' = 3 - 2t$$

$$\int \frac{1}{y^2}\, y'\, dt = \int 3 - 2t\, dt$$

$\frac{1}{y^2} = y^{-2}$

$$\int \frac{1}{y^2}\, dy = \int 3 - 2t\, dt$$

$$\frac{1}{-1} \cdot y^{-1} = 3t - t^2 + C$$ ← implicit solution

$$\frac{1}{y} = -3t + t^2 - C$$

$$y = \frac{1}{-3t + t^2 - C} = \underline{\underline{\frac{1}{t^2 - 3t - C}}}$$