LECTURE 1

GRA 6035

MATHEMATICS

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PLAN:

1. Intro to GRA 6035
2. Linear systems
3. Gaussian elimination
4. Rank of a matrix

Reading:

[LSGE] Notes available for download; Ch 1-3.

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1. For practical information and overview of the course, see Syllabus (available in It's Learning)

* The class will be divided in Group 1-2 before Problem Sessions.
* Check that you know the prerequisites.
* My office hours Wedn. 10-12 in B4-032 is available if you need help or have questions. If you have classes Wedn. 10-12, it is possible to arrange for other office hours, just ask.

② Linear systems

A linear equation in the variables $x_1, x_2, \dots, x_n$ has the form

$$\boxed{a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b}$$

where $a_1, a_2, \dots, a_n, b$ are numbers.

Ex: $x + y = 5$

$x_1 - 2x_2 + 3x_3 = 7$

(linear equation)

$x^2 + y = 4$

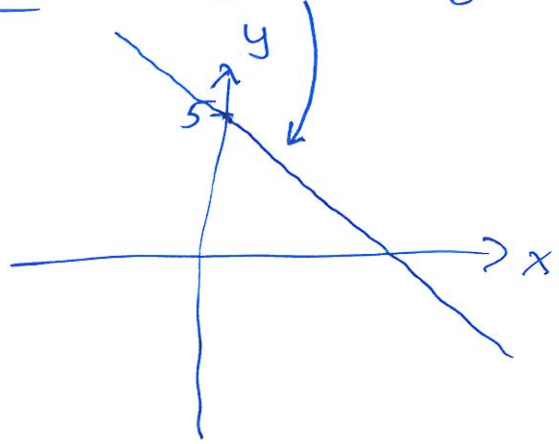
$xy = 1$

$e^x \cdot x = y$

(not linear)

Fact: Linear equations have graphs that are straight lines $(n=2)$, straight planes $(n=3)$ etc.

Ex: $x + y = 5 \iff y = -x + 5$



An $m \times n$ linear system in the variables $x_1, x_2, \dots, x_n$ has the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

where a_{ij}, b_i are numbers.

Ex: (2x2) $\begin{cases} x + y = 2 \\ x - y = 0 \end{cases}$

Ex: (3x3) $\begin{cases} x + y + z = 3 \\ x + 2y + 4z = 7 \\ x + 3y + 9z = 13 \end{cases}$

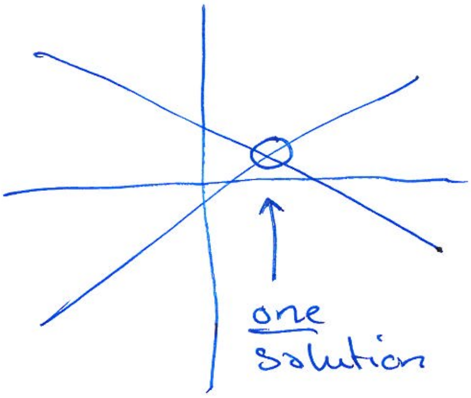
A solution is an n-tuple $(s_1, s_2, \dots, s_n)$ such that $x_1 = s_1, x_2 = s_2, \dots, x_n = s_n$ fits in all the equations.

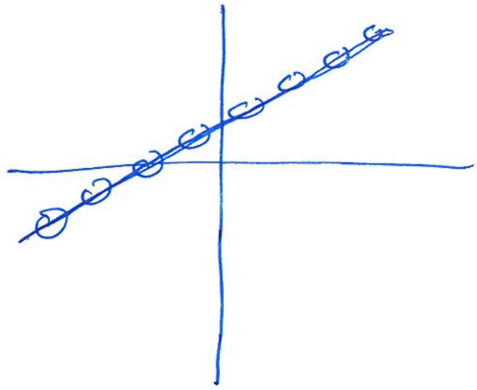
Ex: $x + y = 2$
$x - y = 0$

Solution: $x = 1, y = 1$

Solution types:

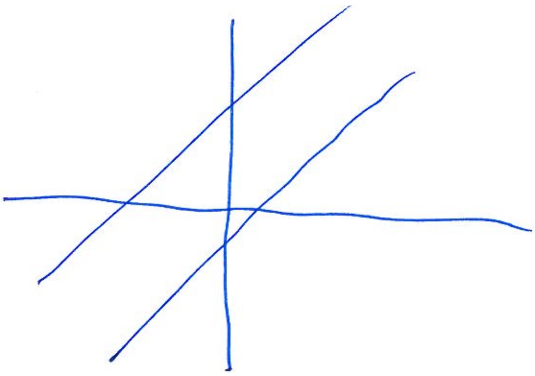
Ex: A 2×2-system
(linear)





double line = both equations have the same line as graph

infinitely many sol's



no solutions

Fact: For any $m \times n$ linear system, ~~exactly one~~ ~~of~~ the following solution types exists:

(1) One unique solution
(2) Infinitely many solution

} system is consistent

(3) No solutions — inconsistent

How do we solve linear systems?

Ex: $\begin{cases} x+y=4 & (1) \\ x-y=2 & (2) \end{cases}$

Substitution	Elimination
(1) $x+y=4$ $\Downarrow$ solve for y $y=4-x$ $\quad$ $y=1$ (2) Substitute for y $x-y=2$ $x-(4-x)=2$ $2x-4=2$ $2x=6$ $x=3$ Solution: $x=3$, $y=1$	$x+y=4$ $x-y=2$ $2x \quad =6$ $x=3$ $y=1$

3 Gaussian elimination

* operate on the system, not the equations

$$\begin{cases} x+y=4 & (1) \\ x-y=2 & (2) \end{cases} \rightarrow \begin{cases} x+y=4 & (1) \\ 2x \quad =6 & (1)+(2) \end{cases}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & -1 & 2 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & 1 & 4 \\ 2 & 0 & 6 \end{array}\right) \quad \begin{array}{l} R(1) \\ R(2) := R(2)+R(1) \end{array}$$

* we use matrices to simplify notation

The coefficient matrix of the linear system is

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The augmented matrix of the linear system is

$$\hat{A} = \left(\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & -1 & 2 \end{array}\right)$$

- Allowed operations: { Operations that do not change the solutions

Elementery row operations:
- are allowed
- are enough to solve all linear systems

- Interchange two rows
- Multiply a row with a number $c \neq 0$

- Change a row by adding to it a number c times another row. $\leftarrow$ $R(i) := R(i) + c \cdot R(j)$

Ex:
$$\begin{aligned} x + 2y &= 4 \\ 3x - y &= 5 \end{aligned} \Rightarrow \begin{array}{c} \begin{matrix} x & y \\ \downarrow & \downarrow \end{matrix} \\ \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & -1 & 5 \end{array}\right) \end{array} \leftarrow R(2) + \underset{\overset{\|}{-3}}{c} \cdot R(1)$$

we want to get 0 in this position

$\downarrow$ elementary row operations

$$\begin{aligned} x + 2y &= 4 \\ -7y &= -7 \end{aligned} \Leftarrow \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -7 & -7 \end{array}\right)$$

back substitution $\rightarrow$ $\underline{y = 1}$ $\quad$ $\underline{x = 2}$

Ex: $$\begin{aligned} x+y+z&=3\\ x+2y+4z&=7\\ x+3y+9z&=13 \end{aligned} \Rightarrow \left(\begin{array}{ccc|c} 1&1&1&3\\ 1&2&4&7\\ 1&3&9&13 \end{array}\right) \begin{array}{l} \\ -1 \\ \\ \end{array}$$

Target: to eliminate as many variables as possible

we want to get zero in these positions.

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1&1&1&3\\ 0&1&3&4\\ 1&3&9&13 \end{array}\right) \begin{array}{l} \\ -1 \\ \\ \end{array}$$

pivot = the first non-zero number in a row

we want zeros under each pivot

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1&1&1&3\\ 0&1&3&4\\ 0&2&8&10 \end{array}\right) \begin{array}{l} \\ \\ -2 \end{array}$$

$$\downarrow$$

echelon form: only zeros under each pivot

$$\left(\begin{array}{ccc|c} 1&1&1&3\\ 0&1&3&4\\ 0&0&2&2 \end{array}\right)$$

$$\begin{aligned} x+y+z&=3\\ y+3z&=4\\ 2z&=2 \end{aligned}$$

Back substitution:

$2z=2 \Rightarrow \underline{z=1}$

$y+3z=4$

$y+3=4 \Rightarrow \underline{y=1}$

$x+y+z=3$

$x+1+1=3 \Rightarrow \underline{x=1}$

Reduced echelon form =

Echelon form such that
- all pivots are 1
- only zeros above all pivots

want zero

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right) :2 \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right) -1$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right) -3 \quad 2 \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

reduced echelon form

$$x = 1$$
$$y = 1$$
$$z = 1$$

Gaussian elimination: stop with any echelon form + back substitution

Gauss-Jordan elimination: stop with reduced echelon form pivot positions: the positions in the augmented matrix where there are pivots when the matrix is in echelon form.

Fact: The reduced echelon form is unique

The pivot positions are unique

However, an echelon form is not unique

Cases with no solutions

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 0 & 7 & 4 \\ 1 & 2 & 10 & 12 \end{array}\right) \begin{matrix} \\ ]-1 \\ \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 0 & 7 & 4 \\ 0 & 0 & 7 & 7 \end{array}\right) \begin{matrix} \\ ]-1 \\ \end{matrix}$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 0 & 7 & 4 \\ 0 & 0 & 0 & 3 \end{array}\right)$$

echelon form

$$x + 2y + 3z = 5$$
$$7z = 4$$
$$0 = 3$$

no solution

In general:

no solution $\Longleftrightarrow$ pivot position in the last column

Cases with infinitely many solutions:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 0 & 7 & 4 \\ 1 & 2 & 10 & 9 \end{array}\right) \xrightarrow{-1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 0 & 7 & 4 \\ 0 & 0 & 7 & 4 \end{array}\right) \xrightarrow{-1}$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 0 & 7 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

echelon form

infinitely many solutions

$\boxed{\begin{array}{l} y \text{ free} \\ x, z \text{ basic} \end{array}}$ (dependent)

In general:

free variable = column without a pivot

basic variable = — " — with a pivot
(dependent)

infinitely many solutions $\Longleftrightarrow$ $\boxed{\text{no pivot in the last column position} + \text{at least one free variable}}$

number of degrees of freedom

= number of free variables

How do we describe the solutions explicitly:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 0 & 7 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x + 2y + 3z = 5$$
$$7z = 4 \Rightarrow z = 4/7$$
$$\cancel{0 = 0}$$

$$x + 2y + 3(4/7) = 5$$
$$x = -2y + 5 - \frac{12}{7} = -2y + \frac{23}{7}$$

$x = 23/7 - 2y$
$y = \text{free}$
$z = 4/7$

solve for dependent (basic) variables

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 23/7 - 2y \\ y \\ 4/7 \end{pmatrix}$$

$$= \begin{pmatrix} 23/7 \\ 0 \\ 4/7 \end{pmatrix} + y \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

Conclusion:

The echelon form has

* pivot position in the last column $\longrightarrow$ no solution

* no pivot position in the last column, but pivot positions in all other columns $\longrightarrow$ one solution

* no pivot position in the last column, and at least one free variable $\longrightarrow$ infinitely many solutions

(4) Rank

The rank of a matrix is the number of pivot positions
It is written rk A.

$$\text{rk } A = \text{number of pivot positions in } A = \text{number of pivots in the echelon form}$$

Ex:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 1 \\ 2 & 4 & 7 \end{pmatrix} \xrightarrow[\text{row operations}]{\quad\cdots\quad} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

echelon form
3 pivots

$$\text{rk } A = 3$$