Now (2) yields

$$\int_0^{10} (1+0.4t)e^{-0.05t}\,dt = \Big|_0^{10} (1+0.4t)(-20)e^{-0.05t} - \int_0^{10} (0.4)(-20)e^{-0.05t}\,dt$$
$$= -100e^{-0.5} + 20 + 8\int_0^{10} e^{-0.05t}\,dt$$
$$= -100e^{-0.5} + 20 - 160(e^{-0.5} - 1) \approx 22.3$$

PROBLEMS FOR SECTION 9.5

SM 1. Use integration by parts to find the following:

(a) $\int xe^{-x}\,dx$ (b) $\int 3xe^{4x}\,dx$ (c) $\int (1+x^2)e^{-x}\,dx$ (d) $\int x\ln x\,dx$

SM 2. Evaluate the following: (a) $\int_{-1}^{1} x\ln(x+2)\,dx$ (b) $\int_0^2 x2^x\,dx$ (c) $\int_0^1 x^2e^x\,dx$

3. Of course, $f(x) = 1 \cdot f(x)$ for any function $f(x)$. Use this fact to prove that

$$\int f(x)\,dx = xf(x) - \int xf'(x)\,dx$$

Apply this formula to $f(x) = \ln x$. Compare with Example 9.1.3.

4. Show that $\int x^\rho \ln x\,dx = \dfrac{x^{\rho+1}}{\rho+1}\ln x - \dfrac{x^{\rho+1}}{(\rho+1)^2} + C, \quad (\rho \neq -1)$.

SM 5. Evaluate the following integrals ($r \neq 0$):

(a) $\int_0^T bte^{-rt}\,dt$ (b) $\int_0^T (a+bt)e^{-rt}\,dt$ (c) $\int_0^T (a-bt+ct^2)e^{-rt}\,dt$

9.6 Integration by Substitution

In this section we shall see how the chain rule for differentiation leads to an important method for evaluating many complicated integrals. We start with some simple examples.

EXAMPLE 1 Evaluate (a) $\int (x^2+10)^{50}2x\,dx$ (b) $\int_0^a xe^{-cx^2}\,dx$ $(c \neq 0)$

Solution:

(a) Attempts to use integration by parts fail. Expanding $(x^2+10)^{50}$ to get a polynomial of 51 terms, and then integrating term by term, would be extremely cumbersome. Instead, let us introduce $x^2 + 10$ as a new variable. We let dx denote the differential of x, and argue as follows: If we substitute $u = x^2 + 10$, then $du = 2x\,dx$, and inserting this into the integral in (a) yields

EXAMPLE 2 Evaluate $\int 8x^2(3x^3-1)^{16}\,dx$.

Solution: Substitute $u = 3x^3 - 1$. Then $du = 9x^2\,dx$, so that $8x^2\,dx = \frac{8}{9}\,du$. Hence

$$\int 8x^2(3x^3-1)^{16}\,dx = \frac{8}{9}\int u^{16}\,du = \frac{8}{9}\cdot\frac{1}{17}u^{17} + C = \frac{8}{153}(3x^3-1)^{17} + C$$

The definite integral in Example 1(b) can be evaluated more simply by "carrying over" the limits of integration. We substituted $u = -cx^2$. As x varies from 0 to a, so u varies from 0 to $-ca^2$. This allows us to write:

$$\int_0^a xe^{-cx^2}\,dx = \int_0^{-ca^2}\frac{-1}{2c}e^u\,du = \frac{-1}{2c}\Big|_0^{-ca^2} e^u = \frac{1}{2c}(1-e^{-ca^2})$$

This method of carrying over the limits of integration can be used in general. In fact,

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \qquad (u = g(x)) \tag{2}$$

The argument is simple: Provided that $F'(u) = f(u)$, we obtain

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \Big|_a^b F\big(g(x)\big) = F\big(g(b)\big) - F\big(g(a)\big) = \int_{g(a)}^{g(b)} f(u)\,du$$

EXAMPLE 3 Evaluate the integral $\displaystyle\int_1^e \frac{1+\ln x}{x}\,dx$.

Solution: We suggest the substitution $u = 1 + \ln x$. Then $du = (1/x)\,dx$. Also, if $x = 1$ then $u = 1$, and if $x = e$ then $u = 2$. So we have

$$\int_1^e \frac{1+\ln x}{x}\,dx = \int_1^2 u\,du = \frac{1}{2}\Big|_1^2 u^2 = \frac{1}{2}(4-1) = \frac{3}{2}$$

PROBLEMS FOR SECTION 9.6

1. Find the following integrals by using (1):

   (a) $\int (x^2+1)^8\,2x\,dx$ (b) $\int (x+2)^{10}\,dx$ (c) $\displaystyle\int \frac{2x-1}{x^2-x+8}\,dx$

SM 2. Find the following integrals by means of an appropriate substitution:

   (a) $\int x(2x^2+3)^5\,dx$ (b) $\int x^2e^{x^3+2}\,dx$ (c) $\displaystyle\int \frac{\ln(x+2)}{2x+4}\,dx$

   (d) $\int x\sqrt{1+x}\,dx$ (e) $\displaystyle\int \frac{x^3}{(1+x^2)^3}\,dx$ (f) $\int x^5\sqrt{4-x^3}\,dx$

3. Find the following integrals:

(a) $\int_0^1 x\sqrt{1+x^2}\,dx$ (b) $\int_1^e \frac{\ln y}{y}\,dy$ (c) $\int_1^3 \frac{1}{x^2}e^{2/x}\,dx$

4. Solve the equation $\int_3^x \frac{2t-2}{t^2-2t}\,dt = \ln\left(\frac{2}{3}x-1\right)$ for values of x satisfying $x > 2$.

5. Show that $\int_{t_0}^{t_1} S'\big(x(t)\big)\dot{x}(t)\,dt = S\big(x(t_1)\big) - S\big(x(t_0)\big)$.

HARDER PROBLEMS

SM 6. Calculate the following integrals:

(a) $\int_0^1 (x^4-x^9)(x^5-1)^{12}\,dx$ (b) $\int \frac{\ln x}{\sqrt{x}}\,dx$ (c) $\int_0^4 \frac{dx}{\sqrt{1+\sqrt{x}}}$

SM 7. Calculate:

(a) $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}\,(1+e^{\sqrt{x}})}\,dx$ (b) $\int_0^{1/3} \frac{dx}{e^x+1}$ (c) $\int_{8.5}^{41} \frac{dx}{\sqrt{2x-1}-\sqrt[4]{2x-1}}$

(*Hint:* For (b), substitute $t = e^{-x}$; for (c), substitute $z^4 = 2x-1$.)

9.7 Infinite Intervals of Integration

In Example 9.6.1(b) we proved that

$$\int_0^a xe^{-cx^2}\,dx = \frac{1}{2c}(1-e^{-ca^2})$$

Suppose c is a positive number and let a tend to infinity. Then the right-hand expression tends to $1/(2c)$. This makes it seem natural to write

$$\int_0^\infty xe^{-cx^2}\,dx = \frac{1}{2c}$$

In statistics and economics it is common to encounter such integrals over an infinite interval.

In general, suppose f is a function that is continuous for all $x \geq a$. Then $\int_a^b f(x)\,dx$ is defined for each $b \geq a$. If the limit of this integral as $b \to \infty$ exists (and is finite), then we say that f is **integrable over** $[a, \infty)$, and define

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx \tag{1}$$

The solution procedure for the general linear differential equation (2) is somewhat more complicated, and we refer to FMEA.

PROBLEMS FOR SECTION 9.9

1. Solve the equation $x^4\dot{x} = 1 - t$. Find the integral curve through $(t, x) = (1, 1)$.

(SM) 2. Solve the following differential equations

(a) $\dot{x} = e^{2t}/x^2$ (b) $\dot{x} = e^{-t+x}$ (c) $\dot{x} - 3x = 18$

(d) $\dot{x} = (1+t)^6/x^6$ (e) $\dot{x} - 2x = -t$ (f) $\dot{x} + 3x = te^{t^2-3t}$

(SM) 3. Suppose that $y = \alpha k e^{\beta t}$ denotes production as a function of capital k, where the factor $e^{\beta t}$ is due to technical progress. Suppose that a constant fraction $s \in (0, 1)$ is saved, and that capital accumulation is equal to savings, so that we have the separable differential equation

$$\dot{k} = s\alpha k e^{\beta t}, \quad k(0) = k_0$$

The constants α, β, and k_0 are positive. Find the solution.

4. Suppose $Y = Y(t)$ is national product, $C(t)$ is consumption at time t, and $\bar{I}$ is investment, which is constant. Suppose $\dot{Y} = \alpha(C + \bar{I} - Y)$ and $C = aY + b$, where a, b, and α are positive constants with $a < 1$.

(a) Derive a differential equation for Y.

(b) Find its solution when $Y(0) = Y_0$ is given. What happens to $Y(t)$ as $t \to \infty$?

(SM) 5. (a) In a growth model production Q is a function of capital K and labour L. Suppose that (i) $\dot{K} = \gamma Q$, (ii) $Q = K^\alpha L$, (iii) $\dot{L}/L = \beta$ with $L(0) = L_0$. Assume that $\beta \neq 0$ and $\alpha \in (0, 1)$. Derive a differential equation for K.

(b) Solve this equation when $K(0) = K_0$.

(SM) 6. Find $x(t)$ when $\mathrm{El}_t\, x(t) = a$ for all t. Assume that both t and x are positive and that a is a constant. (Recall that $\mathrm{El}_t\, x(t)$ is the elasticity of $x(t)$ w.r.t. t.)

REVIEW PROBLEMS FOR CHAPTER 9

1. Find the following integrals:

(a) $\int (-16)\, dx$ (b) $\int 5^5\, dx$ (c) $\int (3 - y)\, dy$ (d) $\int (r - 4r^{1/4})\, dr$

(e) $\int x^8\, dx$ (f) $\int x^2\sqrt{x}\, dx$ (g) $\int \frac{1}{p^5}\, dp$ (h) $\int (x^3 + x)\, dx$

2. Find the following integrals:

(a) $\int 2e^{2x}\, dx$ (b) $\int (x - 5e^{\frac{2}{5}x})\, dx$ (c) $\int (e^{-3x} + e^{3x})\, dx$ (d) $\int \frac{2}{x+5}\, dx$

(SM) 3. Evaluate the following integrals:

(a) $\int_0^{12} 50\,dx$ (b) $\int_0^2 (x - \frac{1}{2}x^2)\,dx$ (c) $\int_{-3}^3 (u+1)^2\,du$

(d) $\int_1^5 \frac{2}{z}\,dz$ (e) $\int_2^{12} \frac{3\,dt}{t+4}\,dt$ (f) $\int_0^4 v\sqrt{v^2+9}\,dv$

4. Find the following integrals:

(a) $\int_1^\infty \frac{5}{x^5}\,dx$ (b) $\int_0^1 x^3(1+x^4)^4\,dx$ (c) $\int_0^\infty \frac{-5t}{e^t}\,dt$

(d) $\int_1^e (\ln x)^2\,dx$ (e) $\int_0^2 x^2\sqrt{x^3+1}\,dx$ (f) $\int_{-\infty}^0 \frac{e^{3z}}{e^{3z}+5}\,dz$

(SM) 5. Find the following integrals:

(a) $\int_0^{25} \frac{1}{9+\sqrt{x}}\,dx$ (b) $\int_2^7 t\sqrt{t+2}\,dt$ (c) $\int_0^1 57x^2\sqrt[3]{19x^3+8}\,dx$

6. Find $F'(x)$ if $F(x) = \int_4^x \left(\sqrt{u} + \frac{x}{\sqrt{u}}\right) du$.

7. With $C(Y)$ as the consumption function, suppose the marginal propensity to consume is $C'(Y) = 0.69$, with $C(0) = 1000$. Find $C(Y)$.

8. In the manufacture of a product, the marginal cost of producing x units is $C'(x) = \alpha e^{\beta x} + \gamma$, with $\beta \neq 0$, and fixed costs are C_0. Find the total cost function $C(x)$

9. Suppose that the demand curve is $f(Q) = 100 - 0.05Q$ and the supply curve is $g(Q) = 10 + 0.1Q$. Find the equilibrium price and quantity. Then calculate the consumer and producer surplus.

(SM) 10. The demand curve is $f(Q) = 50/(Q+5)$ and the supply curve is $g(Q) = 4.5 + 0.1Q$. Find the equilibrium price and quantity. Then calculate the consumer and producer surplus.

(SM) 11. (a) Define f for $t > 0$ by $f(t) = 4\frac{(\ln t)^2}{t}$. Find $f'(t)$ and $f''(t)$.

(b) Find possible local extreme points, and sketch the graph of f.

(c) Calculate the area below the graph of f over the interval $[1, e^2]$.

(SM) 12. Solve the following differential equations:

(a) $\dot{x} = -3x$ (b) $\dot{x} + 4x = 12$ (c) $\dot{x} - 3x = 12x^2$

(d) $5\dot{x} = -x$ (e) $3\dot{x} + 6x = 10$ (f) $\dot{x} - \frac{1}{2}x = x^2$

(SM) 13. Solve the following differential equations:

(a) $\dot{x} = tx^2$ (b) $2\dot{x} + 3x = -15$ (c) $\dot{x} - 3x = 30$

(d) $\dot{x} + 5x = 10t$ (e) $\dot{x} + \frac{1}{2}x = e^t$ (f) $\dot{x} + 3x = t^2$

14. (a) Let $V(x)$ denote the number of litres of fuel left in an aircraft's fuel tank if it has flown x km. Suppose that $V(x)$ satisfies the following differential equation:

$$V'(x) = -aV(x) - b$$

(The fuel consumption per km is a constant $b > 0$. The term $-aV(x)$, with $a > 0$, is due to the weight of the fuel.) Find the solution of the equation with $V(0) = V_0$.

(b) How many km, x^*, can the plane fly if it takes off with V_0 litres in its tank?

(c) What is the minimum number of litres, V_m, needed at the outset if the plane is to fly $\hat{x}$ km?

(d) Put $b = 8$, $a = 0.001$, $V_0 = 12\,000$, and $\hat{x} = 1200$. Find x^* and V_m in this case.

15. With reference to "Income Distribution" in Section 9.4, in a population of n individuals let the income distribution function be $f(r) = (1/m)e^{-r/m}$, $r \in [0, \infty)$, where m is a positive constant.

(a) Show that m is the mean income.

(b) Suppose the demand function is $D(p, r) = ar - bp$. Compute the total demand $x(p)$ when the income distribution is as in (a).

SM 16. A probability density function f is defined for all x by

$$f(x) = \frac{\lambda a e^{-\lambda x}}{(e^{-\lambda x} + a)^2} \qquad (a \text{ and } \lambda \text{ are positive constants})$$

(a) Show that $F(x) = \dfrac{a}{e^{-\lambda x} + a}$ is an indefinite integral of $f(x)$, and determine $\lim_{x\to\infty} F(x)$ and $\lim_{x\to-\infty} F(x)$.

(b) Show that $\int_{-\infty}^{x} f(t)\,dt = F(x)$, and that $F(x)$ is strictly increasing.

(c) Compute $F''(x)$ and show that F has an inflection point x_0. Compute $F(x_0)$ and sketch the graph of F.

(d) Compute $\int_{-\infty}^{\infty} f(x)\,dx$.

9.4

1. $x(t) = K - \int_0^t \bar{u}e^{-as}\,ds = K - \bar{u}(1 - e^{-at})/a$. Note that $x(t) \to K - \bar{u}/a$ as $t \to \infty$. If $K \geq \bar{u}/a$, the well will never be exhausted.

2. (a) $m = 2b \ln 2$. (The relevant number of individuals is $n \int_b^{2b} Br^{-2}\,dr = nB/2b$, whose total income is $M = n \int_b^{2b} Br^{-1}\,dr = nB \ln 2$.) (b) $x(p) = \int_b^{2b} nAp^{\gamma} r^{\delta} Br^{-2}\,dr = nABp^{\gamma} b^{\delta-1}(2^{\delta-1} - 1)/(\delta - 1)$

3. (a) $K(5) - K(0) = \int_0^5 (3t^2 + 2t + 5)\,dt = 175$ (b) $K(T) - K_0 = (T^3 - t_0^3) + (T^2 - t_0^2) + 5(T - t_0)$

4. $T = \frac{1}{r} \ln(1 + rS)$. ($S = \Big|_0^T (1/r)e^{rt} = (e^{rT} - 1)/r$, so $e^{rT} - 1 = rS$, and solve for T.)

5. (a) See Fig. A9.4.5. (b) $\int_0^t \big(g(\tau) - f(\tau)\big)\,d\tau = \frac{1}{2}t^2(t-10)^2 \geq 0$ for all t. (c) $\int_0^{10} p(t)f(t)\,dt = 940 + 11 \ln 11 \approx 966.38$, $\int_0^{10} p(t)g(t)dt = 3980/3 - 121 \ln 11 \approx 1036.52$. Profile g should be chosen.

6. The equilibrium quantity is $Q^* = 600$, where $P^* = 80$. Then, $\text{CS} = \int_0^{600} (120 - 0.2Q)\,dQ = 36\,000$, and $\text{PS} = \int_0^{600} (60 - 0.1Q)\,dQ = 18\,000$.

7. Equilibrium when $6000/(Q^* + 50) = Q^* + 10$. The only positive solution is $Q^* = 50$, and then $P^* = 60$.

$$\text{CS} = \int_0^{50} \left[\frac{6000}{Q + 50} - 60\right] dQ = \Big|_0^{50} [6000 \ln(Q+50) - 60Q] = 6000 \ln 2 - 3000, \quad \text{PS} = \int_0^{50} (50 - Q)\,dQ = 1250$$

9.5

1. (a) Use (1) with $f(x) = x$ and $g'(x) = e^{-x}$: $\int xe^{-x}\,dx = x(-e^{-x}) - \int 1 \cdot (-e^{-x})\,dx = -xe^{-x} - e^{-x} + C$.
(b) $\frac{3}{4}xe^{4x} - \frac{3}{16}e^{4x} + C$ (c) $-x^2e^{-x} - 2xe^{-x} - 3e^{-x} + C$ (d) $\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$.

2. (a) $\int_{-1}^1 x \ln(x + 2)\,dx = \Big|_{-1}^1 \frac{1}{2}x^2 \ln(x + 2) - \int_{-1}^1 \frac{1}{2}x^2 \frac{1}{x+2}\,dx = \frac{1}{2} \ln 3 - \frac{1}{2} \int_{-1}^1 \left(x - 2 + \frac{4}{x+2}\right) dx = 2 - \frac{3}{2} \ln 3$
(b) $8/(\ln 2) - 3/(\ln 2)^2$ (c) $e - 2$

3. The general formula follows from (1), and $\int \ln x\,dx = x \ln x - x + C$.

4. Use (1) with $f(x) = \ln x$ and $g'(x) = x^{\rho}$. (Alternatively, simply differentiate the right-hand side.)

5. (a) $br^{-2}\big[1 - (1 + rT)e^{-rT}\big]$ (b) $ar^{-1}(1 - e^{-rT}) + br^{-2}\big[1 - (1 + rT)e^{-rT}\big]$
(c) $ar^{-1}\big(1 - e^{-rT}\big) - br^{-2}\big[1 - (1 + rT)e^{-rT}\big] + cr^{-3}\big[2\big(1 - e^{-rT}\big) - 2rTe^{-rT} - r^2T^2e^{-rT}\big]$

9.6

1. (a) $\frac{1}{9}(x^2 + 1)^9 + C$. (Substitute $u = x^2 + 1$, $du = 2x\,dx$.) (b) $\frac{1}{11}(x + 2)^{11} + C$. (Substitute $u = x + 2$.)
(c) $\ln|x^2 - x + 8| + C$. (Substitute $u = x^2 - x + 8$.)

2. (a) $\frac{1}{24}(2x^2 + 3)^6 + C$. (Substitute $u = 2x^2 + 3$, so $du = 4x\,dx$. The integral becomes $\frac{1}{4} \int u^5\,du$.)
(b) $\frac{1}{3}e^{x^3+2} + C$. (Substitute $u = e^{x^3+2}$.) (c) $\frac{1}{4}\big(\ln(x + 2)\big)^2 + C$. (Substitute $u = \ln(x + 2)$.)
(d) $\frac{2}{5}(1 + x)^{5/2} - \frac{2}{3}(1 + x)^{3/2} + C$. (Substitute $u = \sqrt{1 + x}$.) (e) $\dfrac{-1}{2(1 + x^2)} + \dfrac{1}{4(1 + x^2)^2} + C$.
(f) $\frac{2}{15}(4 - x^3)^{5/2} - \frac{8}{9}(4 - x^3)^{3/2} + C$.

3. (a) Substitute $u = \sqrt{1 + x^2}$. Then $u^2 = 1 + x^2$, so $u\,du = x\,dx$. If $x = 0$, $u = 1$; if $x = 1$, $u = \sqrt{2}$. Hence, $\int_0^1 x\sqrt{1 + x^2}\,dx = \int_1^{\sqrt{2}} u^2\,du = \Big|_1^{\sqrt{2}} \frac{1}{3}u^3 = \frac{1}{3}(2\sqrt{2} - 1)$. (b) 1/2. (Let $u = \ln y$.) (c) $\frac{1}{2}(e^2 - e^{2/3})$. (Let $u = 2/x$.)

4. $\displaystyle\int_3^x \frac{2t - 2}{t^2 - 2t}\,dt = \Big|_3^x \ln(t^2 - 2t) = \ln(x^2 - 2x) - \ln 3 = \ln \tfrac{1}{3}(x^2 - 2x) = \tfrac{2}{3}x - 1$. Hence, $x^2 - 4x + 3 = 0$, with solutions $x = 1$ and $x = 3$. But only $x = 3$ is in the specified domain. So the solution is $x = 3$.

5. Substitute $z = x(t)$. Then $dz = \dot{x}(t)dt$, and the result follows using (2).

6. (a) $1/70$. ($(x^4 - x^9)(x^5 - 1)^{12} = -x^4(x^5 - 1)^{13}$.) (b) $2\sqrt{x}\ln x - 4\sqrt{x} + C$. (Let $u = \sqrt{x}$.) (c) $8/3$

7. (a) $2\ln(1 + e^2) - 2\ln(1 + e)$ (b) $\ln 2 - \ln(e^{-1/3} + 1)$ (c) $7 + 2\ln 2$

9.7

1. (a) $\int_1^b x^{-3}\,dx = \Big|_1^b(-\frac{1}{2}x^{-2}) = \frac{1}{2} - \frac{1}{2}b^{-2} \to \frac{1}{2}$ as $b \to \infty$. So $\int_1^\infty (1/x^3)\,dx = \frac{1}{2}$.
(b) $\int_1^b x^{-1/2}\,dx = \Big|_1^b 2x^{1/2} = 2b^{1/2} - 2 \to \infty$ as $b \to \infty$, so the integral diverges.
(c) 1 (d) $\int_0^a (x/\sqrt{a^2 - x^2})\,dx = -\Big|_0^a \sqrt{a^2 - x^2} = a$

2. (a) $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = \frac{1}{b-a}\Big|_a^b x = \frac{1}{b-a}(b-a) = 1$
(b) $\displaystyle\int_{-\infty}^{+\infty} xf(x)\,dx = \frac{1}{b-a}\int_a^b x\,dx = \frac{1}{2(b-a)}\Big|_a^b x^2 = \frac{1}{2(b-a)}(b^2 - a^2) = \frac{1}{2}(a+b)$
(c) $\displaystyle\frac{1}{3(b-a)}\Big|_a^b x^3 = \frac{1}{3}\frac{b^3 - a^3}{b-a} = \frac{1}{3}(a^2 + ab + b^2)$

3. Using a simplified notation and the result in Example 1, we have:
(a) $\int_0^\infty x\lambda e^{-\lambda x}\,dx = -\Big|_0^\infty xe^{-\lambda x} + \int_0^\infty e^{-\lambda x}\,dx = 1/\lambda$ (b) $1/\lambda^2$ (c) $2/\lambda^3$

4. The first integral diverges because $\int_0^b [x/(1 + x^2)]\,dx = \Big|_0^b \frac{1}{2}\ln(1 + x^2) = \frac{1}{2}\ln(1 + b^2) \to \infty$ as $b \to \infty$. On the other hand, $\int_{-b}^b [x/(1 + x^2)]\,dx = \Big|_{-b}^b \frac{1}{2}\ln(1 + x^2) = 0$ for all b, so the limit as $b \to \infty$ is 0.

5. (a) f has a maximum at $(e^{1/3}, 1/3e)$, but no minimum. (b) $\int_0^1 x^{-3}\ln x\,dx$ diverges. $\int_1^\infty x^{-3}\ln x\,dx = 1/4$.

6. $1/(1 + x^2) \le 1/x^2$ for $x \ge 1$, and $\int_1^b (1/x^2)\,dx = \Big|_1^b(-1/x) = 1 - 1/b \to 1$ as $b \to \infty$, so by Theorem 9.7.1 the integral converges.

7. $4\sqrt{5}$

8. (a) $z = \int_0^\tau (1/\tau)e^{-rs}\,ds = (1 - e^{-r\tau})/r\tau$ (b) $z = \int_0^\tau 2(\tau - s)\tau^{-2}e^{-rs}ds = (2/r\tau)\big[1 - (1/r\tau)(1 - e^{-r\tau})\big]$

9. $\int x^{-2}dx = -x^{-1} + C$. So evaluating $\int_{-1}^1 x^{-2}dx$ as $\Big|_{-1}^1 - x^{-1}$ gives the nonsensical answer -2. The error arises because x^{-2} diverges to $+\infty$ as $x \to 0$. (In fact $\int_{-1}^1 x^{-2}dx$ diverges to $+\infty$.)

10. Using the answer to Problem 9.6.6(b), $\int_h^1 (\ln x/\sqrt{x})\,dx = \Big|_h^1(2\sqrt{x}\ln x - 4\sqrt{x}) = -4 - (2\sqrt{h}\ln h - 4\sqrt{h}) \to -4$ as $h \to 0^+$, so the given integral converges to -4. ($\sqrt{h}\ln h = \ln h/h^{-1/2} \to 0$, by l'Hôpital's rule.)

11. $\int_1^A [k/x - k^2/(1 + kx)]\,dx = k\ln[1/(1/A + k)] - k\ln[1/(1 + k)] \to k\ln(1/k) - k\ln[1/(1 + k)] = \ln(1 + 1/k)^k$ as $A \to \infty$. So $I_k = \ln(1 + 1/k)^k$, which tends to $\ln e = 1$ as $k \to \infty$.

12. The proofs are given in SM.

9.8

1. The functions in (c) and (d) are the only ones which have a constant relative rate of increase. This accords with (3). (Note that $2^t = e^{(\ln 2)t}$.)

2. (a) $K(t) = (K_0 - I/\delta)e^{-\delta t} + I/\delta$ (b) (i) $K(t) = 200 - 50e^{-0.05t}$ and $K(t)$ tends to 200 from below as $t \to \infty$. (ii) $K(t) = 200 + 50e^{-0.05t}$, and $K(t)$ tends to 200 from above as $t \to \infty$.

3. $N(t) = P(1 - e^{-kt})$. Then $N(t) \to P$ as $t \to \infty$.

4. $\dot{N}(t) = 0.02N(t) + 4 \cdot 10^4$. The solution with $N(0) = 2 \cdot 10^6$ is $N(t) = 2 \cdot 10^6(2e^{0.02t} - 1)$.

5. $k = 0.1\ln(705/641) \approx 0.0095$. $P(15) \approx 739$, $P(40) \approx 938$

6. The percentage surviving after t seconds satisfies $p(t) = 100e^{-\delta t}$, where $p(7) = 70.5$ and so $\delta = -\ln 0.705/7 \approx 0.05$. Thus $p(30) = 100e^{-30\delta} \approx 22.3\%$ are still alive after 30 seconds. Because $100e^{-\delta t} = 5$ when $t \approx \ln 20/0.05 \approx 60$, it takes about 60 seconds to kill 95%.

7. (a) $x = Ae^{-0.5t}$ (b) $K = Ae^{0.02t}$ (c) $x = Ae^{-0.5t} + 10$ (d) $K = Ae^{0.2t} - 500$ (e) $x = 0.1/(3 - Ae^{0.1t})$ (f) $K = 1/(2 - Ae^{t})$

8. (a) $y(t) = 250 + \dfrac{230}{1 + 8.2e^{-0.34t}}$. (b) $y(t) \to 480$ as $t \to \infty$. See Fig. A9.8.8.

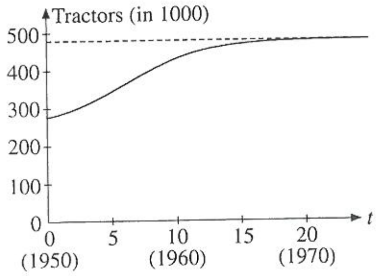

Figure A9.8.8

9. (a) Using (7) we find $N(t) = 1000/(1 + 999e^{-0.39t})$. After 20 days, $N(20) = 710$ have developed influenza. (b) Approximately 21 days. (c) After about 35 days, 999 will have or have had influenza. $N(t) \to 1000$ as $t \to \infty$.

10. At about 11:26. (Measuring time in hours, with $t = 0$ being 12 noon, one has $\dot{T} = k(20 - T)$ with $T(0) = 35$ and $T(1) = 32$. So the body temperature at time t is $T(t) = 20 + 15e^{-kt}$ with $k = \ln(5/4)$. Assuming that the temperature was the normal 37 degrees at the time of death t^*, then $t^* = -\ln(17/15)/\ln(5/4) \approx -0.56$ hours, or about 34 minutes before 12:00.)

9.9

1. General solution: $x = \sqrt[5]{5t - \frac{5}{2}t^2 + C}$. $x(1) = 1$ yields $C = -3/2$. (The equation is separable: $\int x^4\,dx = \int (1-t)\,dt$, $\frac{1}{5}x^5 = t - \frac{1}{2}t^2 + C_1$, $x^5 = 5t - \frac{5}{2}t^2 + 5C_1$, $x = \sqrt[5]{5t - \frac{5}{2}t^2 + 5C_1}$, and put $C = 5C_1$.)

2. (a) $x = \sqrt[3]{\frac{3}{2}e^{2t} + C}$ (b) $x = -\ln(e^{-t} + C)$ (c) $x = Ce^{3t} - 6$ (d) $x = \sqrt[7]{(1+t)^7 + C}$ (e) $x = Ce^{2t} + \frac{1}{2}t + \frac{1}{4}$ (f) $x = Ce^{-3t} + \frac{1}{2}e^{t^2-3t}$

3. $k = k_0 e^{(s\alpha/\beta)(e^{\beta t}-1)}$

4. (a) $\dot{Y} = \alpha(a-1)Y + \alpha(b + \bar{I})$ (b) $Y = \left(Y_0 - \dfrac{b+\bar{I}}{1-a}\right)e^{-\alpha(1-a)t} + \dfrac{b+\bar{I}}{1-a} \to \dfrac{b+\bar{I}}{1-a}$ as $t \to \infty$.

5. (a) From (iii), $L = L_0e^{\beta t}$, so $\dot{K} = \gamma K^{\alpha} L_0 e^{\beta t}$, a separable equation. (b) $K = \left[\frac{(1-\alpha)\gamma}{\beta}L_0(e^{\beta t} - 1) + K_0^{1-\alpha}\right]^{1/(1-\alpha)}$

6. $x(t) = At^a$ where A is an arbitrary constant.

Review Problems for Chapter 9

1. (a) $-16x + C$ (b) $5^5x + C$ (c) $3y - \frac{1}{2}y^2 + C$ (d) $\frac{1}{2}r^2 - \frac{16}{5}r^{5/4} + C$ (e) $\frac{1}{9}x^9 + C$ (f) $\frac{2}{7}x^{7/2} + C$ (g) $-\frac{1}{4}p^{-4} + C$ (h) $\frac{1}{4}x^4 + \frac{1}{2}x^2 + C$

2. (a) $e^{2x} + C$ (b) $\frac{1}{2}x^2 - \frac{25}{2}e^{\frac{2}{5}} + C$ (c) $-\frac{1}{3}e^{-3x} + \frac{1}{3}e^{3x} + C$ (d) $2\ln|x+5| + C$

3. (a) 600 (b) 2/3 (c) 24 (d) $2\ln 5$ (e) $3\ln(8/3)$ (f) 98/3

4. (a) 5/4 (see Example 9.7.2). (b) $\Big|_0^1 \frac{1}{20}(1+x^4)^5 = 31/20$ (c) $\Big|_0^\infty 5te^{-t} - \int_0^\infty 5e^{-t}\,dt = 5\Big|_0^\infty e^{-t} = -5$
(d) $\int_1^e (\ln x)^2\,dx = \Big|_1^e x(\ln x)^2 - 2\int_1^e \ln x\,dx = e - 2\Big|_1^e(x\ln x - x) = e - 2$
(e) $\Big|_0^2 \frac{2}{9}(x^3+1)^{3/2} = \frac{2}{9}(9^{3/2} - 1) = 52/9$ (f) $\Big|_{-\infty}^0 \frac{1}{3}\ln(e^{3z}+5) = \frac{1}{3}(\ln 6 - \ln 5) = \frac{1}{3}\ln(6/5)$

5. (a) $10 - 18\ln(14/9)$. (Substitute $z = 9 + \sqrt{x}$.) (b) 886/15. (Substitute $z = \sqrt{t+2}$.) (c) 195/4. (Substitute $z = \sqrt[3]{19x^3+8}$.)

6. $F'(x) = 4(\sqrt{x} - 1)$. $\left(\int_4^x (u^{1/2} + xu^{-1/2})\,du = \Big|_4^x \frac{2}{3}u^{3/2} + 2xu^{1/2} = \frac{8}{3}x^{3/2} - \frac{16}{3} - 4x.\right)$

7. $C(Y) = 0.69Y + 1000$ **8.** $C(x) = \frac{\alpha}{\beta}(e^{\beta x} - 1) + \gamma x + C_0$

9. $P^* = 70$, $Q^* = 600$, $CS = 9000$, $PS = 18\,000$ **10.** $P^* = Q^* = 5$, $\mathrm{CS} = 50\ln 2 - 25$, $\mathrm{PS} = 1.25$

11. (a) $f'(t) = 4\ln t(2 - \ln t)/t^2$, $f''(t) = 8[(\ln t)^2 - 3\ln t + 1]/t^3$. (b) $(e^2, 16/e^2)$ is a local maximum point, $(1, 0)$ is a local (and global) minimum point. See Fig. A9.R.11. (c) Area = 32/3. (*Hint:* $\int f(t)\,dt = \frac{4}{3}(\ln t)^3 + C$.)

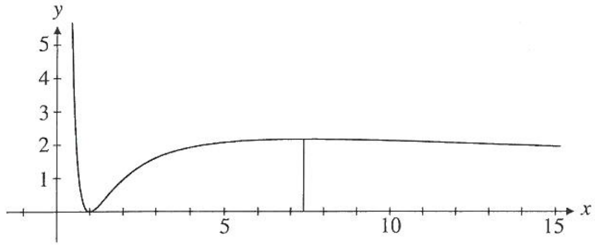

Figure A9.R.11

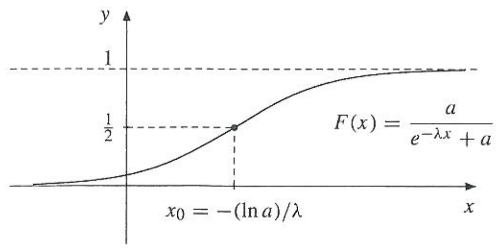

Figure A9.R.16

12. (a) $x = Ae^{-3t}$ (b) $x = Ae^{-4t} + 3$ (c) $x = 1/(Ae^{-3t} - 4)$ and $x \equiv 0$. (d) $x = Ae^{-\frac{1}{5}t}$ (e) $x = Ae^{-2t} + 5/3$ (f) $x = 1/(Ae^{-\frac{1}{2}t} - 2)$ and $x \equiv 0$.

13. (a) $x = 1/(C - \frac{1}{2}t^2)$ and $x(t) \equiv 0$. (b) $x = Ce^{-3t/2} - 5$ (c) $x = Ce^{3t} - 10$ (d) $x = Ce^{-5t} + 2t - \frac{2}{5}$
(e) $x = Ce^{-t/2} + \frac{2}{3}e^t$ (f) $x = Ce^{-3t} + \frac{1}{3}t^2 - \frac{2}{9}t + \frac{2}{27}$

14. (a) $V(x) = (V_0 + b/a)e^{-ax} - b/a$ (b) $V(x^*) = 0$ yields $x^* = (1/a)\ln(1 + aV_0/b)$.
(c) $0 = V(\hat{x}) = (V_m + b/a)e^{-a\hat{x}} - b/a$ yields $V_m = (b/a)(e^{a\hat{x}} - 1)$.
(d) $x^* = (1/0.001)\ln(1 + 0.001 \cdot 12\,000/8) \approx 916$, and $V_m = (8/0.001)(e^{0.001\cdot 1200} - 1) = 8000(e^{1.2} - 1) \approx 18561$.

15. (a) $\int_0^\infty f(r)\,dr = \int_0^\infty (1/m)e^{-r/m}\,dr = 1$ (as in Example 9.7.1) and $\int_0^\infty rf(r)\,dr = \int_0^\infty r(1/m)e^{-r/m}\,dr = m$ (as in Problem 9.7.3(a)), so the mean income is m.
(b) $x(p) = n\int_0^\infty (ar - bp)f(r)\,dr = n(a\int_0^\infty rf(r)\,dr - bp\int_0^\infty f(r)\,dr = n(am - bp)$

16. (a) Verify that $F'(x) = f(x)$. $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$.
(b) $\int_{-\infty}^x f(t)\,dt = \lim_{a\to-\infty}\int_a^x f(t)\,dt = \lim_{a\to-\infty}[F(x) - F(a)] = F(x)$, by (a). Since $F'(x) = f(x) > 0$ for all x, $F(x)$ is strictly increasing. For (c) and (d) see Fig. A9.R.16 and SM.

Chapter 10

10.1

1. (a) (i) $8000(1 + 0.05/12)^{5\cdot 12} \approx 10266.87$ (ii) $8000(1 + 0.05/365)^{5\cdot 365} \approx 10272.03$
(b) $t = \ln 2/\ln(1 + 0.05/12) \approx 166.7$. It takes approximately $166.7/12 \approx 13.9$ years.

2. (a) $5000(1 + 0.03)^{10} \approx 6719.58$ (b) 37.17 years. ($5000(1.03)^t = 3 \cdot 5000$, so $t = \ln 3/\ln 1.03 \approx 37.17$.)