Problem Sheet 8 with Solutions
GRA 6035 Mathematics

BI Norwegian Business School

Problems

1. [FMEA] Problem 3.1.4.

2. [FMEA] Problem 3.1.5.

3. [FMEA] Problem 3.3.4.

4. [EMEA] Problem 3.5.1.

5. [EMEA] Problem 3.5.3.

6. Final Exam in GRA6035 10/12/2010, Problem 4
We consider the function $f(x,y,z) = xyz$.

a) The function g is defined on the set $D = \{(x,y,z) : x > 0, y > 0, z > 0\}$, and it is given by

$$g(x,y,z) = \frac{1}{f(x,y,z)} = \frac{1}{xyz}$$

Is g a convex or concave function on D?

b) Maximize $f(x,y,z)$ subject to $x^2 + y^2 + z^2 \leq 1$.

7. Mock Final Exam in GRA6035 12/2010, Problem 4
We consider the following optimization problem: Maximize $f(x,y,z) = xy + yz - xz$ subject to the constraint $x^2 + y^2 + z^2 \leq 1$.

a) Write down the first order conditions for this problem, and solve the first order conditions for x,y,z using matrix methods.

b) Solve the optimization problem. Make sure that you check the non-degenerate constraint qualification, and also make sure that you show that the problem has a solution.

8. Final Exam in GRA6035 30/05/2011, Problem 4
We consider the function $f(x,y) = xye^{x+y}$ defined on $D_f = \{(x,y) : (x+1)^2 + (y+1)^2 \leq 1\}$.

a) Compute the Hessian of f. Is f a convex function? Is f a concave function?

b) Find the maximum and minimum values of f.

Solutions

For Problem 1-5, see the solutions in [EG] Eriksen, Gustavsen (Problems 7.1-7.3 and 8.10-8.11).

6 Final Exam in GRA6035 10/12/2010, Problem 4

a) We compute the Hessian of g, and find

$$g'' = \frac{1}{xyz}\begin{pmatrix} \frac{2}{x^2} & \frac{1}{xy} & \frac{1}{xz} \\ \frac{1}{xy} & \frac{2}{y^2} & \frac{1}{yz} \\ \frac{1}{xz} & \frac{1}{yz} & \frac{2}{z^2} \end{pmatrix}$$

Hence the leading principal minors are

$$D_1 = \frac{1}{xyz}\frac{2}{x^2} > 0, \quad D_2 = \frac{1}{(xyz)^2}\frac{3}{(xy)^2} > 0, \quad D_3 = \frac{1}{(xyz)^3}\frac{4}{(xyz)^2} > 0$$

This means that g is convex.

b) The set $\{(x,y,z) : x^2+y^2+z^2 \leq 1\}$ is closed and bounded, so the problem has solutions by the extreme value theorem. The NDCQ is satisfied, since the rank of $\begin{pmatrix}2x & 2y & 2z\end{pmatrix} = 1$ when $x^2+y^2+z^2 = 1$. We form the Lagrangian

$$\mathscr{L} = xyz - \lambda(x^2+y^2+z^2-1)$$

and solve the Kuhn-Tucker conditions, consisting of the first order conditions

$$\begin{aligned} \mathscr{L}'_x &= yz - \lambda \cdot 2x = 0 \\ \mathscr{L}'_y &= xz - \lambda \cdot 2y = 0 \\ \mathscr{L}'_z &= xy - \lambda \cdot 2z = 0 \end{aligned}$$

together with one of the following conditions: i) $x^2+y^2+z^2 = 1$ and $\lambda \geq 0$ or ii) $x^2+y^2+z^2 < 1$ and $\lambda = 0$. We first solve the equations/inequalities in case i): If $x = 0$, then we see that $y = 0$ or $z = 0$ from the first equation, and we get the solutions $(x,y,z;\lambda) = (0,0,\pm 1;0), (0,\pm 1,0;0)$. If $x \neq 0$, we get $2\lambda = yz/x$ and the remaining first order conditions give $(x^2-y^2)z = 0$ and $(x^2-z^2)y = 0$. If $y = 0$, we get the solution $(\pm 1,0,0;0)$. Otherwise, we get $x^2 = y^2 = z^2$, hence

$$(x,y,z;\lambda) = \left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}; \pm\frac{1}{2\sqrt{3}}\right)$$

The condition that $\lambda \geq 0$ give that either all three coordinates (x,y,z) are positive, or that one is positive and two are negative. In total, we obtain four different solutions. We note that $f(x,y,z) = \frac{1}{3\sqrt{3}}$ for each of these four solutions, while $f(x,y,z) = 0$ for either of the first three solutions. Finally, we consider case ii), where $\lambda = 0$. This gives $xy = xz = yz = 0$, and we obtain

$$(x,y,z;\lambda) = (a,0,0;0),(0,a,0;0),(0,0,a;0)$$

The condition that $x^2+y^2+z^2<1$ give $a^2 \le 1$ or $a \in (-1,1)$. For all these solutions, we get $f(x,y,z) = 0$. We can therefore conclude that the solution to the optimization problem is a maximum value of

$$\mathbf{\frac{1}{3\sqrt{3}}}$$

7 Mock Final Exam in GRA6035 12/2010, Problem 4

See handwritten solution on the coarse page for GRA 6035 Mathematics 2010/11.

8 Final Exam in GRA6035 30/05/2011, Problem 4

a) We compute the Hessian of f, and find

$$f'' = e^{x+y}\begin{pmatrix} (x+2)y & (x+1)(y+1) \\ (x+1)(y+1) & x(y+2) \end{pmatrix}$$

The principal minors are

$$\Delta_1 = e^{x+y}(x+2)y,\ \Delta_1 = e^{x+y}x(y+2), \quad D_2 = (e^{x+y})^2(1-(x+1)^2-(y+1)^2)$$

Since $(x+1)^2+(y+1)^2 \le 1$, D_f is a ball with center in $(-1,-1)$ and radius $r=1$, and it follows that $x,y<0$ and $x+2,y+2 \ge 0$, and therefore $\Delta_1 \le 0$ and $D_2 \ge 0$. This means that f is concave, but not convex.

b) Since D_f is closed and bounded, f has maximum and minimum values. We compute the stationary points of f: We have

$$f'_x = (x+1)ye^{x+y} = 0, \quad f'_y = x(y+1)e^{x+y} = 0$$

and $(x,y) = (0,0)$ and $(x,y) = (-1,-1)$ are the solutions. Hence there is only one stationary point $(x,y) = (-1,-1)$ in D_f, and the $f(-1,-1) = \mathbf{e^{-2}}$ is the maximum value of f since f is concave. The minimum value most occur for (x,y) on the boundary of D_f. We see that $f(x,y) \ge 0$ for all $(x,y) \in D_f$ while $f(-1,0) = f(0,-1) = 0$. Hence $\mathbf{f(-1,0) = f(0,-1) = 0}$ is the minimum value of f.