Question 6 /Mid-term 28.09.09 (lecture notes, pag 210)

Consider the matrix

$$A=\begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & -10 \\ 0 & 0 & -2 \end{pmatrix}.$$

The matrix has the eigenvectors

$$u=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ v=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ w=\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

where u and v have the eigenvalue λ=8 and w has the eigenvalue λ= -2. Which statement is correct?

A. The matrix A does not have three distinct eigenvalues. Hence it is not diagonalizable.
B. The matrix A does not have three linearly independent eigenvectors, and it is not diagonalizable.
C. The matrix A is diagonalizable.
D. The matrix A is not invertible.
E. I prefer not to answer.

Remember:

1) **If A is a $n_x n$ matrix with n different eigenvalues, then A is diagonalizable.**

!!! (If A is diagonalizable, its eigenvalues are not necessary different)

2) **A is diagonalizable if and only if there are n linearly independent eigenvectors for A.**

Solution:

The vectors u, v and w are linearly independent, since the matrix that has them as column vectors (matrix B) has the determinant 1 (different from 0).

Det (B)= det $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$=1

We found 3 linearly independent eigenvectors, so we can conclude that A is diagonalizable.

Deeply to understand:

- we know the eigenvalues and we compute the eigenvectors corresponding to them;

  ---for λ=8: Ax=8x → (A-8 I)x=0 → $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -10 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x1 \\ x2 \\ x3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. We find that x1 and x2 are free variables and x3=0 .

Therefore, the eigenvectors corresponding to λ=8 are: r* $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s * \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, where r and s are numbers.

If we take for instance r=1 and s=0, we obtain u=$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

If we take r=0 and s=1, we obtain v= $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

--- for **λ=-2:** Ax= -2x → (A-*(-2)* I)x=0 → $\begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & -10 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x1 \\ x2 \\ x3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. We find the solution $\begin{pmatrix} x1 \\ x2 \\ x3 \end{pmatrix}=s * \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, where s is a number.

The matrix A does not have three distinct eigenvalues. Even if we found 2 eigenvalues that are equal ($\lambda_1= \lambda_2=8$), we were able to find 3 linearly independent eigenvectors for matrix A (u,v and w). Hence matrix A is diagonalizable.