Lecture 8
Differential Equations

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October 22, 2010

Review: Kuhn-Tucker conditions

To review Kuhn-Tucker conditions, we shall start by solving the following optimization problem with inequality constraints:

Example

Maximize $f(x, y) = xy + x^2$ *subject to* $x^2 + y \leq 2$ *and* $y \geq 1$.

Solution

The full solution will be given during Lecture 8. The short answer is that $f(1, 1) = 2$ *is the maximum.*

Ordinary differential equations

We consider functions $y = f(t)$ in one variable:

Ordinary differential equations

An ordinary differential equation or ODE is an equation involving the variable t, the function $y = f(t)$ and one or more of its derivatives $y' = f'(t)$, $y'' = f''(t)$, $\ldots$. Examples of ODE's are

- $y' = 2t + 3$
- $y' = 4y - 1$
- $y'' - 2y' + y = t^2 + 1$

The order of an ordinary differential equation is the highest order of the derivatives that appear in the differential equation.

It is also possible to consider partial differential equations for functions in more than one variable. This is a more advanced topic. We will only cover ordinary differential equations of order one and two in this course.

Differential equations: Motivation

When $y = y(t)$ is an economic parameter that is time-dependent, economic models that describe the changes in y over time will lead to differential equations in y.

Example

According to an economic model, the growth rate of $y = y(t)$ is proportional with $y(t)$ with a factor of proportionality of 5.3. This might be expressed by the differential equation

$$y'(t) = 5.3\, y(t) \quad \text{or} \quad y' = 5.3y$$

In applications, differential equations are often complemented by initial conditions or start conditions. For instance, the initial condition could be that $y(0) = 10$ in the above model. This means that $y = 10$ when $t = 0$.

Solutions of differential equations

Definition

A solution of a differential equation is a function $y = f(t)$ such that the the equation holds when we replace y and its derivatives with $f(t)$ and its derivatives.

Here is one example:

Example

Consider the differential equation $y' = 2y$. Then $f(t) = e^{2t}$ is a solution, since we have $y' = f'(t) = 2e^{2t}$ and $2y = 2f(t) = 2e^{2t}$. Moreover, we see that

$$f(t) = Ce^{2t} \quad \Rightarrow \quad f'(t) = 2f(t)$$

so $f(t) = Ce^{2t}$ is a solution for any constant C.

Differential equations

General solution and particular solutions

Definition (General solution)

The set of all solutions of a differential equation is called the general solution. It can often be described as a function $y = f(t)$, where $f(t)$ is a function that depends on parameters.

General fact: An ordinary differential equation of order n has a general solution $f(t)$ that depends on n parameters.

Definition (Particular solutions)

A specific function that solves a differential equation is called a particular solution. It is often obtained by giving the parameters in the general solution $y = f(t)$ particular numerical values.

Differential equations: Examples

Example

The general solution of the first order differential equation $y' = 2y$ *is* $f(t) = Ce^{2t}$*. When we replace* C *with a specific numerical value such as* $C = 1$*, we obtain the various particular solutions such as* $f(t) = e^{2t}$*.*

An initial value problem is a differential equation together with an initial condition. This solution is a particular solution:

Example

Consider the initial value problem

$$y' = 2y, \quad y(0) = 4$$

The general solution of the differential equation $y' = 2y$ *is* $f(t) = Ce^{2t}$*. The initial condition is* $4 = Ce^{2\cdot 0} \Leftrightarrow C = 4$*, so the solution of the initial value problem is the particular solution* $f(t) = 4e^{2t}$*.*

ODE's solvable by direct integration

The simplest type of differential equations have the form $y' = a(t)$, and can be solved by integration:

$$y' = a(t) \quad \Rightarrow \quad y = \int a(t)\, dt$$

Example

The differential equation $x' = 12t - 1$ *is a first order ordinary differential equation, and a solution is a function* $x = f(t)$ *such that* $f'(t) = 12t - 1$*. In this case, we may solve the differential equation by integration:*

$$f'(t) = 12t - 1 \quad \Rightarrow \quad f(t) = \int (12t - 1)\, dt = 6t^2 - t + \mathcal{C}$$

The general solution is therefore $f(t) = 6t^2 - t + \mathcal{C}$*.*

Review: Integration

As the previous example shows, integration is essential in the process of solving differential equations.

Example

Compute the integrals

- $\int x^{13}\, dx$
- $\int (t^3 + 2t - 3)\, dt$
- $\int xe^x\, dx$
- $\int x(x^2+1)^8\, dx$
- $\int \ln t\, dt$

For those who need to review integration further, it could be a good idea to review Chapter 9 from [EMEA] or a similar text.

First order ODE's

A first order ODE is a differential equation that contains t, y and y' but no higher order derivatives. We often write a first order ODE in the form

$$y' = F(y,t)$$

where $F(y,t)$ is a function in y and t. Most differential equations are impossible to solve. We shall consider the following kinds of first order ODE's:

1. Linear: $y' + a(t)y = b(t)$
2. Separable: $y' = a(y)b(t)$
3. Exact: $a(y,t) + b(y,t)y' = 0$ where $\frac{\partial a}{\partial y} = \frac{\partial b}{\partial t}$

In Lecture 8 and Lecture 9 we shall see how we can solve these kinds of ODE's.

Separable differential equations

Definition

A first order ODE is separable if it can be written in the form $y' = a(y)b(t)$, where $a(y)$ is a function in y and $b(t)$ is a function in t.

Example

The differential equation $y' = yt$ is separable, since it can be written as $y' = a(y)b(t)$, with $a(y) = y$ and $b(t) = t$. The differential equation $y' = yt + 1$ is not separable, since $yt + 1$ cannot be factorized as $a(y)b(t)$.

To solve a separable differential equation, we must separate it in the following way:

$$y' = a(y)b(t) \quad \Leftrightarrow \quad \frac{1}{a(y)}\, y' = b(t) \quad \Leftrightarrow \quad \frac{1}{a(y)}\, dy = b(t)\, dt$$

When the differential equation has been separated, it can be solved by integration.

Separable ODE's: An example

Example

Solve the differential equation $y' = yt$.

Solution

We separate the differential equation:

$$y' = yt \quad \Leftrightarrow \quad \frac{1}{y}\, y' = t \quad \Leftrightarrow \quad \frac{1}{y}\, dy = t\, dt$$

Integration of both sides gives $\int \frac{1}{y}\, dy = \int t\, dt$, and therefore

$$\ln|y| = \frac{1}{2}t^2 + \mathcal{C}' \Rightarrow |y| = \exp\left(\frac{1}{2}t^2 + \mathcal{C}'\right) \Rightarrow y = C \exp\left(\frac{1}{2}t^2\right)$$

We use the notation $\exp(x) = e^x$ for any x.

Separability

Example

Are the following ODE's separable?

- $y' = y + t$
- $y' = yt + 2t$
- $y' = yt^2 + y^2t^2$
- $3y^2y' = 2t$

Solution

- $y' = y + t$ *is not separable*
- $y' = yt + 2t \quad \Rightarrow \quad y' = (y+2)t$ *is separable*
- $y' = yt^2 + y^2t^2 \quad \Rightarrow \quad y' = (y + y^2)t^2$ *is separable*
- $3y^2y' = 2t \quad \Rightarrow \quad y' = \frac{1}{3y^2}2t$ *is separable*

Separable ODE's: Another example

Example

Find the general solution of the differential equation $3y^2y' = 2t$, and the particular solution that satisfy the initial condition $y(0) = -1$.

Solution

We separate the differential equation:

$$3y^2y' = 2t \quad \Leftrightarrow \quad 3y^2\,dy = 2t\,dt$$

Integration of both sides gives $\int 3y^2\,dy = \int 2t\,dt$, and therefore

$$y^3 = t^2 + \mathcal{C} \Rightarrow y = \sqrt[3]{t^2 + \mathcal{C}}$$

To solve the initial condition $y(0) = -1$, we set $t = 0$ and $y = -1$ into the general solution. It is easiest to use the implicit form $y^3 = t^2 + \mathcal{C}$, which gives $\mathcal{C} = -1$ and solution $y = \sqrt[3]{t^2 - 1}$.

Separable ODE's: A more difficult example

Example

Solve the differential equation $x' = x(1-x)$.

Solution

The differential equation in $x = x(t)$ *is separable, and we get*

$$x' = x(1-x) \Leftrightarrow \int \frac{1}{x(1-x)}\,dx = \int 1\,dt$$

To solve the first integral, we find the partial fractions

$$\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$$

When we compute the integrals, we therefore get

$$\ln|x| - \ln|1-x| = t + \mathcal{C}'$$

Separable ODE's: A more difficult example

Solution (Continued)

We simplify and get

$$\ln\left|\frac{x}{1-x}\right| = t + \mathcal{C}' \quad \Leftrightarrow \quad \left|\frac{x}{1-x}\right| = \exp(t + \mathcal{C}')$$

When we remove the absolute value, we obtain

$$\frac{x}{1-x} = \mathcal{C}e^t \quad \Leftrightarrow \quad x = (1-x)\mathcal{C}e^t \quad \Leftrightarrow \quad x(1+\mathcal{C}e^t) = \mathcal{C}e^t$$

This gives general solution in explicit form

$$x = \frac{\mathcal{C}e^t}{1+\mathcal{C}e^t}$$