Plan

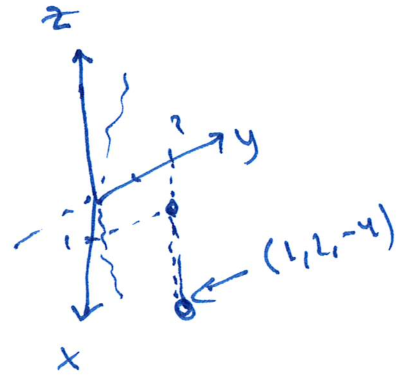


## 1 Functions of two variables

Ex: $f(x,y) = 1 - x^2 - y^2$

$D_f = \{(x,y) : (x,y) \text{ in } \mathbb{R}^2\}$

$f(1,2) = 1 - 1 - 4 = \underline{-4}$




Level curves: $z = c$

$f(x,y) = c$

Graph:
$z = f(x,y)$

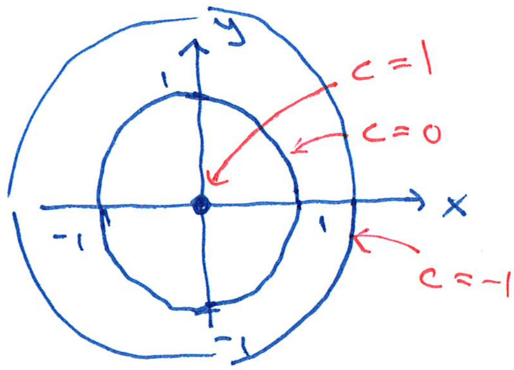
Ex: $f(x,y) = 1 - x^2 - y^2$

Level curve: $f(x,y) = c$

$1 - x^2 - y^2 = c$

$c = 0$: $1 - x^2 - y^2 = 0$

circle, $r = 1$, center $(0,0)$ $\longrightarrow x^2 + y^2 = 1$




$c = 1$: $1 - x^2 - y^2 = 1$

$x^2 + y^2 = 0$ pt. $(0,0)$

$c = -1$: $1 - x^2 - y^2 = -1$

$x^2 + y^2 = 2$

Circle, $r = \sqrt{2}$, center $(0,0)$

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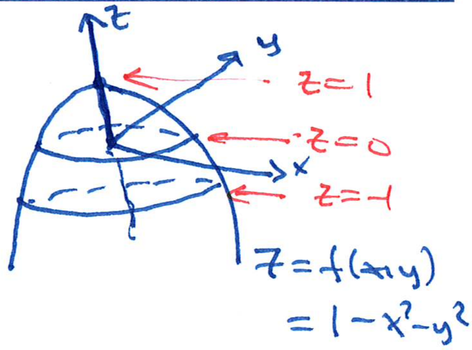
Putting the level curves together:




Partial derivatives:

$$f'_x = f'_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

slope of the tangent line to $z=f(x,y)$ when $y$ is fixed

$$f'_y = f'_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

slope of the tangent line to $z=f(x,y)$ when $x$ is fixed

Ex: $f(x,y) = 1-x^2-y^2$

$$f'_x = 0-2x-0 = \underline{-2x} = f'_x(x,y)$$

$$f'_y = 0-0-2y = \underline{-2y} = f'_y(x,y)$$

Ex: $f(x,y) = e^{1-x^2-y^2} = e^u, \quad u = 1-x^2-y^2$

$$f'_x = (e^u)'_u \cdot (1-x^2-y^2)'_x = e^u \cdot (-2x) = \underline{-2x \cdot e^{1-x^2-y^2}}$$

$$f'_y = (e^u)'_u \cdot (1-x^2-y^2)'_y = e^u \cdot (-2y) = \underline{-2y \cdot e^{1-x^2-y^2}}$$

$$f'_x(1,2) = -2 \cdot e^{-4} = -2/e^4$$

$$f'_y(1,2) = -4 \cdot e^{-4} = -4/e^4$$

## ② Unconstrained optimization

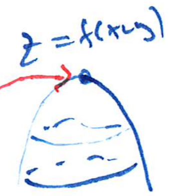
max/min $f(x,y)$

**Defn:** A <u>stationary pt</u> for $f$ is a point where $\boxed{f'_x = f'_y = 0}$ ← FOC = first order conditions

**Fact:** If $(x,y) = (a,b)$ is a max or min for $f$, then this pt. has to be a stationary pt for $f$, unless
* $(a,b)$ is a <u>boundary</u> pt for $D_f$
* $f'_x(a,b)$ or $f'_y(a,b)$ does not exist

**Ex:** $f(x,y) = 1 - x^2 - y^2$

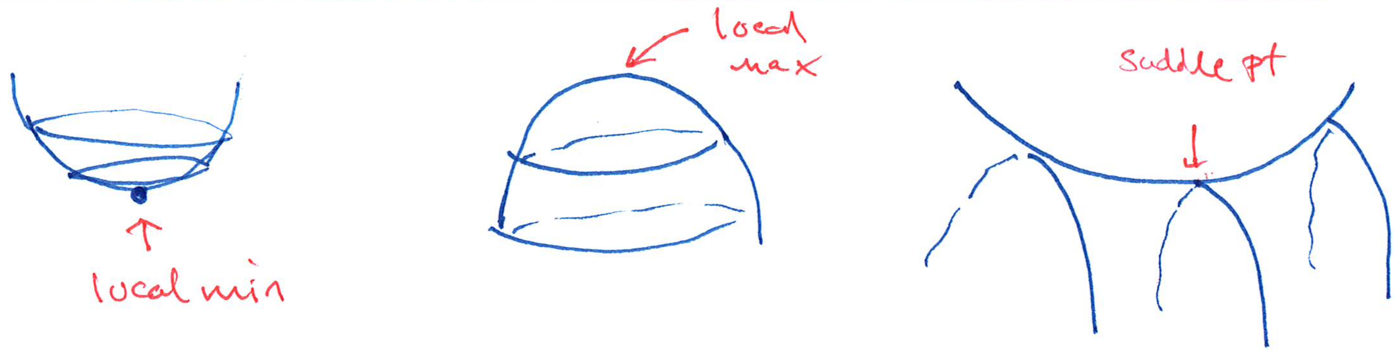
$$\left.\begin{aligned} f'_x &= -2x = 0 \qquad x = 0 \\ f'_y &= -2y = 0 \qquad y = 0 \end{aligned}\right\} \quad \underline{\text{Stationary pts:}} \ (x,y) = (0,0)$$




**Defn:** A pt. $(x,y) = (a,b)$ is a <u>local max</u> for $f$ if $f(a,b) \geq f(x,y)$ for all pts $(x,y)$ close to $(a,b)$.

— " — <u>local min</u> — " — if $f(a,b) \leq f(x,y)$
— " — close to $(a,b)$.

A stationary pt that is neither local max nor local min is called a <u>saddle pt</u>.




## Second derivative test:

Let $(a,b)$ be a stationary pt of $f$, and compute

$$H(f)(a,b) = \begin{pmatrix} f''_{xx}(a,b) & f''_{xy}(a,b) \\ f''_{yx}(a,b) & f''_{yy}(a,b) \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

Hessian matrix of $f$ at $(a,b)$.

Fact: For all "nice" functions, $f''_{xy} = f''_{yx} \Leftrightarrow H(f)$ is a symmetric matrix

Thm:

If $AC-B^2>0$ and $A>0$ then $(a,b)$ is a local min

If $AC-B^2>0$ and $A<0$ — " — a local max

If $AC-B^2<0$ — " — a saddle pt

Note: If $AC-B^2=0$ then the test is inconclusive

Note: If $AC-B^2>0$ then $AC>B^2\geq 0 \Rightarrow$ $A>0, C>0$ or $A<0, C<0$

$AC-B^2 = \det H(f)(a,b)$

Ex: $f(x,y) = 1 - x^2 - y^2$

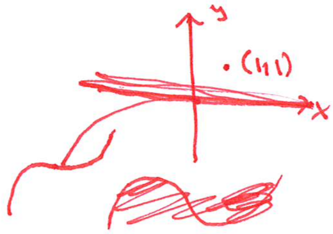
$f'_x = \underline{-2x} = 0 \qquad f''_{xx} = -2 \qquad f''_{xy} = 0$

$f'_y = \underline{-2y} = 0 \qquad f''_{yx} = 0 \qquad f''_{yy} = -2$

Stationary pt: $\underline{(0,0)}$ $\qquad H(f)(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

$AC - B^2 = \underline{4} > 0 \qquad A = \underline{-2} < 0$

$\Rightarrow$ $(0,0)$ is a <u>local max</u> for $f$ by the second derivative test




$y = 0$

$f(x,0) = x^3$

Ex: $f(x,y) = x^3 - 3xy + y^3$

$f'_x = \underline{3x^2 - 3y} = 0 \qquad f''_{xx} = 6x \qquad f''_{xy} = -3$

$f'_y = \underline{-3x + 3y^2} = 0 \qquad f''_{yx} = -3 \qquad f''_{yy} = 6y$

$\frac{3x^2}{3} = \frac{3y}{3} \qquad \underline{y = x^2}$

$-3x + 3(x^2)^2 = 0$

$-3x + 3x^4 = 0$

$3x(-1 + x^3) = 0$

$\underline{x=0}$, $\underline{y=0}$ or $x^3 = 1$, $\underline{x = \sqrt[3]{1} = 1}$, $\underline{y = 1}$

<u>Stationary pts:</u> $(0,0)$, $(1,1)$

<u>$(0,0)$:</u> $\underline{H(f)(0,0)} = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$ $\quad \det = 0 - (-3)^2 = -9 < 0$

$\Rightarrow$ $(0,0)$ is <u>saddle pt</u>

<u>$(1,1)$:</u> $H(f)(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$ $\quad \det = 36 - 9 = 27 > 0 \quad A > 0$

$\Rightarrow$ $(1,1)$ is a <u>local min</u>

<u>Concl:</u> <u>no max, no min</u> $\qquad f(1,1) = -1 \qquad f(-2,0) = -8 < -1$

**Ex:** $f(x,y) = e^{1-x^2-y^2}$ $\qquad e^u > 0$ for any $u$

$f'_x = -2x \cdot e^{1-x^2-y^2} = 0 \qquad -2x = 0 \qquad x = 0$

$f'_y = -2y \cdot e^{1-x^2-y^2} = 0 \qquad -2y = 0 \qquad y = 0$

Stat pts: $(0,0)$

$$f''_{xx} = (-2x \cdot e^u)'_x = -2 \cdot e^u + (-2x) \cdot e^u \cdot u'_x$$

$$= -2e^u + (-2x) \cdot e^u \cdot (-2x)$$

$$= -2e^u + 4x^2 e^u = (4x^2 - 2)e^{1-x^2-y^2}$$

$f''_{xx}(0,0) = -2e$ $\qquad \bullet$

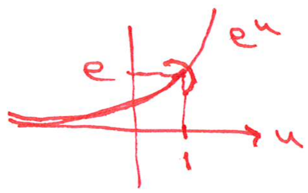
$\bullet \qquad \bullet$

....

**Better way:** $f(x,y) = e^{1-x^2-y^2} = e^u, \quad u = 1 - x^2 - y^2$

$e^u$ is increasing in $u$

max for $u$: $u(0,0) = 1$

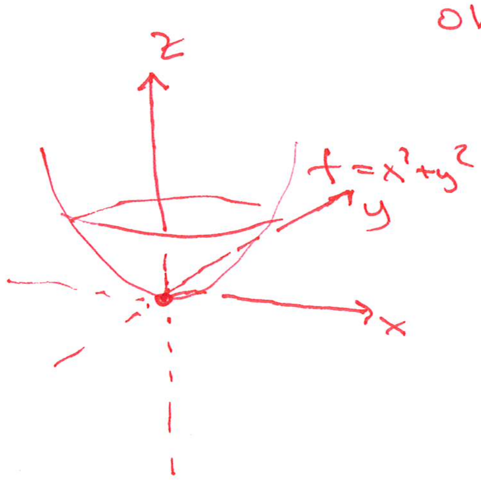


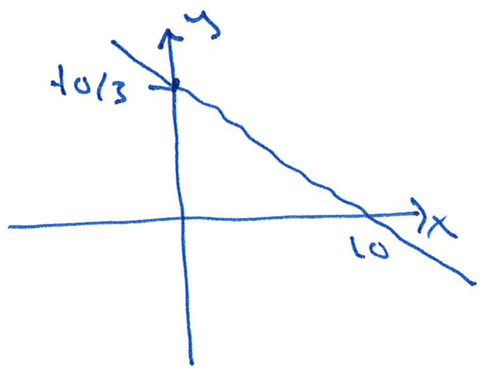

$(e^u)' = e^u > 0$

**Concl:** max of $f$ is $f(0,0) = e$

## ③ Constrained optimization and Lagrange multipliers

Ex: max/min $\underbrace{f(x,y) = x^2+y^2}_{\text{Objective fn.}}$ when $\underbrace{x+3y=10}_{\text{constraint}}$




$x+3y=10$

$3y = -x+10$

$y = -\frac{1}{3}x + \frac{10}{3}$

### Method of Lagrange multipliers:

$$L(x,y;\lambda) = f(x,y) - \lambda\cdot(x+3y-10)$$

$$= x^2+y^2 - \lambda(x+3y-10)$$

Lagrangian

(FOC) $\begin{cases} L'_x = 2x - \lambda\cdot 1 = 0 \\ L'_y = 2y - \lambda\cdot 3 = 0 \end{cases}$

(C) $L'_\lambda = -1(x+3y-10) = 0 \quad | \cdot (-1) \rightarrow x+3y-10=0$

$x+3y=10$

Solutions of FOC+C:

$(x,y;\lambda) = \underline{(1,3;2)}$

= Candidate points

(FOC) $2x - \lambda = 0 \qquad x = \lambda/2 = 1$

$2y - 3\lambda = 0 \qquad y = 3\lambda/2 = 3$

(C) $x+3y=10 \qquad \lambda/2 + 3\cdot(3\lambda/2) = 10 \quad |\cdot 2 \qquad \lambda + 9\lambda = 20 \qquad \frac{\lambda = 2}{10\lambda = 20}$

**Fact:** If $(x,y)=(a,b)$ is a max/min in the problem

$$\max/\min f(x,y) \quad \text{when} \quad g(x,y)=c$$

then ~~$L'_x(a,b)=0$, $L'_y(a,b)=0$~~ and $g(a,b)=c$.

$L'_x(a,b;\lambda)=0, \quad L'_y(a,b;\lambda)=0$

for some $\lambda$, unless $g'_x(a,b)=g'_y(a,b)=0$ and $g(a,b)=c$.

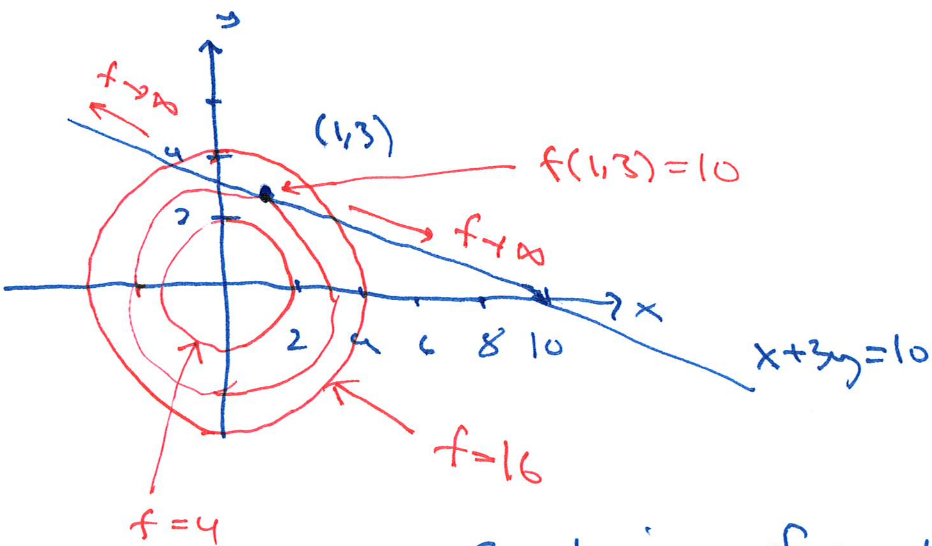
**Note:**

Exceptions: $g(x,y)=x+3y$

$$\left.\begin{array}{l} g'_x=1=0 \\ g'_y=3=0 \end{array}\right\} \text{impossible} \Rightarrow \text{no exceptions}$$

**Concl:** $(x,y)=(1,3)$ with $\lambda=2$ is the only possibility for max/min.

$f(1,3)=10$




Level curve for f:

$f(x,y)=10$

$x^2+y^2=10$

$f(x,y)=c>10$

$x^2+y^2=c$

$f(x,y)=c<10$

Conclusion: $f_{min}=10$ at $(1,3)$