Plan

1. Key Method: Determinants
2. Linear systems, inverse matrices and determinants

Review:

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

2×2-matrix $(n=2)$

Facts:

1) $|A| = ad-bc$

2) If $|A| \neq 0$, then $A^{-1} = \frac{1}{|A|}\cdot\begin{pmatrix} d & -b \\ -c & a\end{pmatrix}$

If $|A| = 0$, then A is not invertible (A^{-1} does not exist)

Lin. Sys:

$ax + by = e$

$cx + dy = f$

Gauss: $a \neq 0$

$$\left(\begin{array}{cc|c} a & b & e \\ c & d & f\end{array}\right) \xrightarrow{-c/a} \left(\begin{array}{cc|c} a & b & e \\ 0 & d-\frac{bc}{a} & f-\frac{ec}{a}\end{array}\right)\cdot a$$

$$\to \left(\begin{array}{cc|c} a & b & e \\ 0 & ad-bc & af-ec\end{array}\right)$$

$ad-bc \neq 0$

One solution

$ax + by = e$

$\cancel{ax} + (ad-bc)y = af-ec$

$y = \dfrac{af-ec}{ad-bc}$

$x = \dfrac{e-by}{a}$

$= \dfrac{e}{a} - \dfrac{b}{a}\cdot\dfrac{af-ec}{ad-bc}$

$ad-bc = 0$

$af-ec \neq 0$: no solutions

$af-ec = 0$: y free, inf. many solutions

Fact:

$\boxed{|A| \neq 0 \Updownarrow Ax = b \text{ has exactly one solution}}$

$|A| \neq 0$:

$A\underline{x} = \underline{b} \quad | A^{-1}\cdot$

$A^{-1}A\underline{x} = A^{-1}\underline{b}$

$\underline{x} = A^{-1}\underline{b}$

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$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \cdot \begin{pmatrix} e \\ f \end{pmatrix}$$

$$= \frac{1}{ad-bc} \begin{pmatrix} de-bf \\ -ce+af \end{pmatrix} \qquad x = \frac{de-bf}{ad-bc}$$

$$y = \frac{-ce+af}{ad-bc}$$

1 Key method: Determinants

A $\rightsquigarrow$ $\det(A) = |A|$

$n \times n$-matrix (square) — a number

$\boxed{n=2: \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad |A| = ad-bc}$

Method: Cofactor expansion

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 6 \end{pmatrix}$

3×3-matrix ($n=3$)

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 6 \end{vmatrix} = 1 \cdot C_{11} + 1 \cdot C_{12} + 1 \cdot C_{13}$$

$$= \underbrace{1 \cdot (+1) \cdot \begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix}}_{C_{11}} + \underbrace{1 \cdot (-1) \cdot \begin{vmatrix} 1 & 4 \\ 2 & 6 \end{vmatrix}}_{C_{12}}$$

$$+ \underbrace{1 \cdot (+1) \cdot \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}}_{C_{13}}$$

$$= +1 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 4 \\ 2 & 6 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$$

$$= +1\,(2\cdot 6 - 3\cdot 4) - 1\,(1\cdot 6 - 4\cdot 2) + 1\cdot(1\cdot 3 - 2\cdot 2) = 0 + 2 - 1 = \underline{\underline{1}}$$

C_{ij}: cofactor in position (i,j)

$= (-1)^{i+j} \cdot M_{ij}$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Minor in pos (i,j) = determinant of the submatrix you get by deleting row i, col. j.

Ex:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 6 \end{vmatrix} = -1\cdot\begin{vmatrix} 1 & 4 \\ 2 & 6 \end{vmatrix} + 2\cdot\begin{vmatrix} 1 & 1 \\ 2 & 6 \end{vmatrix} - 3\cdot\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix}$$

$$= -(-2) + 2(4) - 3(3) = 2 + 8 - 9 = \underline{\underline{1}}$$

Fact: Cofactor expansion along any row or column of a matrix A gives the same result, $\underline{\det(A)}$.

Ex:

$$\begin{vmatrix} 1 & 7 & 13 \\ 0 & 4 & -11 \\ 0 & 0 & 2 \end{vmatrix} = +1\cdot\begin{vmatrix} 4 & -11 \\ 0 & 2 \end{vmatrix} = 1\cdot 4\cdot 2 = \underline{\underline{8}}$$

The case $n=3$:

$$\left|\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}\right| \begin{matrix} a & b \\ d & e \\ g & h \end{matrix} = aei + bfg + cdh - ceg - afh - bdi$$

$$\|$$

$$+a(ei - fh) - b(di - fg) + c\cdot(dh - eg)$$

$$= aei - afh - bdi + bfg + cdh - ceg$$

Cofactor expansion is a general method (can be used for any n)

Ex:

$$\begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 4 & 3 & 0 \\ 0 & 3 & 4 & 0 \\ -1 & 0 & 0 & 1 \end{vmatrix} = +1 \cdot \begin{vmatrix} 4 & 3 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 1 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 0 & 4 & 3 \\ 0 & 3 & 4 \\ -1 & 0 & 0 \end{vmatrix}$$

$$= +\big(+1 \cdot (16-9)\big) + 1\big(-1(16-9)\big) = 7 - 7 = \underline{\underline{0}}$$

Method: Using Gaussian elimination

Ex:

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 4 & 3 & 0 \\ 0 & 3 & 4 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\;1\;} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 4 & 3 & 0 \\ 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;{-3/4}$$

$$4 - \frac{9}{4} = \frac{16-9}{4} = \frac{7}{4}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 4 & 3 & 0 \\ 0 & 0 & 7/4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = E$$

$$|E| = 1 \cdot 4 \cdot 7/4 \cdot 0 = \underline{\underline{0}}$$

$$\Rightarrow |A| = |E| = \underline{\underline{0}}$$

Defn: A square matrix is upper triangular if all entries below / to the left of the main diagonal are zero.

Facts:
- Echelon forms are always upper triangular. (of square matrices)
- Determinant of an upper triangular matrix is the product of the diagonal entries.

Let $A \to B$ be one elementary row operation

Thm: $|B| = |A|$ if we add a multiple of one row to another row

$|B| = -|A|$ if we switch two rows

$|B| = c \cdot |A|$ if we multiply a row by $c \neq 0$

Fact: $|A| = 0$ if either
i) A has a zero row
ii) A has two equal rows
iii) A has a row that is a multiple of another row

Ex: $\begin{vmatrix} 1 & 7 & 4 \\ 0 & 3 & 5 \\ 1 & 7 & 4 \end{vmatrix} = 0$

② Inverse matrices

A $n \times n$-matrix (square)

Fact: A is invertible $\Leftrightarrow$ $|A| \neq 0$
(A^{-1} exists)

The linear system $A\underline{x} = \underline{b}$ has one unique solution $\Leftrightarrow$ $|A| \neq 0$

$$\underline{x} = A^{-1}\underline{b}$$

$\underline{n=2}$: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\qquad A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

if $ad - bc \neq 0$

General formula:

A $n \times n$-matrix with $|A| \neq 0$: $\boxed{A^{-1} = \frac{1}{|A|} \cdot \operatorname{adj}(A)}$

adjungated matrix of A

transpose

$$= \frac{1}{|A|} \cdot \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{pmatrix}^T$$

cofactor matrix of A

$$= \frac{1}{|A|} \cdot \begin{pmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 6 \end{pmatrix}$ $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

$C_{11} = 0 \quad C_{12} = 2 \quad C_{13} = -1$

$C_{21} = -3 \quad C_{22} = 4 \quad C_{23} = -1$

$C_{31} = 2 \quad C_{32} = -3 \quad C_{33} = 1$

cofactor matrix: $C = \begin{pmatrix} 0 & 2 & -1 \\ -3 & 4 & -1 \\ 2 & -3 & 1 \end{pmatrix}$

adj matrix: $\text{adj}(A) = C^T = \begin{pmatrix} 0 & -3 & 2 \\ 2 & 4 & -3 \\ -1 & -1 & 1 \end{pmatrix}$

determinant: $|A| = 1 \cdot 0 + 1 \cdot 2 + 1 \cdot (-1)$
$= 0 + 2 - 1 = \underline{1} \neq 0$

$A^{-1} = \frac{1}{|A|} \cdot \text{adj}(A) = \frac{1}{1} \cdot \begin{pmatrix} 0 & -3 & 2 \\ 2 & 4 & -3 \\ -1 & -1 & 1 \end{pmatrix}$

$$A^{-1} = \begin{pmatrix} 0 & -3 & 2 \\ 2 & 4 & -3 \\ -1 & -1 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 3 & 6 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 0 & -3 & 2 \\ 2 & 4 & -3 \\ -1 & -1 & 1 \end{pmatrix}$$

$$A \cdot A^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ +2 & -3 & 6 \end{pmatrix} \cdot \begin{pmatrix} 0 & -3 & 2 \\ 2 & 4 & -3 \\ -1 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$C = \begin{pmatrix} 0 & 2 & -1 \\ -3 & 4 & -1 \\ 2 & -3 & 1 \end{pmatrix}$$

$$\underline{1} \cdot \underline{\underline{0}} + \underline{1} \cdot \underline{\underline{2}} + \underline{1} \cdot (\underline{\underline{-1}}) = |A| \leftarrow$$ cofactor expansion along the first row of A

<u>Alt method</u> (Lecture 2):

$$(A \mid I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 2 & 3 & 6 & 0 & 0 & 1 \end{array}\right) \to \dots \to$$ reduced echelon form

if $(I \mid C)$ then $A^{-1} = C$