	Plan
1	Integration and basic integration rules
2	Integration techniques

Cont'd from Lecture 3:

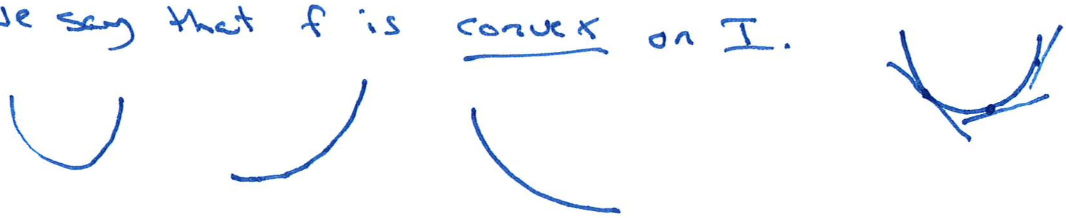
(3) Higher derivatives and convexity

Ex: $f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$

$f''(x) = (3x^2-3)' = 6x$ second derivative

Defn: If $f''(x) \geq 0$ for all x in an interval I, we say that f is convex on I.

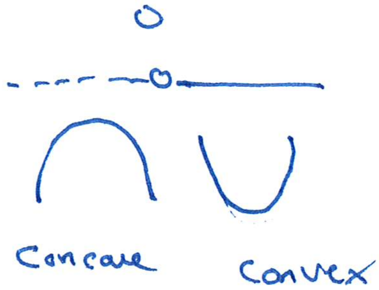


If $f''(x) \leq 0$ for all x in an interval I, we say that f is concave on I

Ex: $f(x) = x^3 - 3x + 2$

$f''(x) = 6x$

$f'' = 6x$

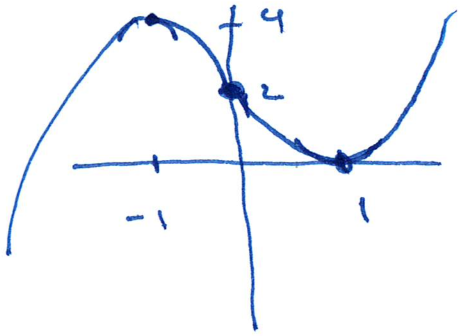


Fact:

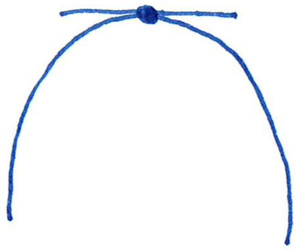
convex everywhere

$I = D_f$

If f is convex, then any stationary pt of f is a <u>min</u>



If f is concave, then any stationary pt of f is a <u>max</u>



Ex: $f(x) = 0.6 \ln(1+x) + 0.4 \ln(1-x)$, $0 \le x < 1$

$$f'(x) = \underbrace{\frac{0.6}{1+x} - \frac{0.4}{1-x}} = \frac{0.2 - x}{(1+x)(1-x)}$$

$$f''(x) = \left(0.6 \cdot \underline{(1+x)^{-1}} - 0.4(1-x)^{-1}\right)'$$

$$= 0.6 \cdot (-1) \cdot (1+x)^{-2} \cdot 1 - 0.4(-1) \cdot (1-x)^{-2} \cdot (-1)$$

$$= -\frac{0.6}{(1+x)^2} - \frac{0.4}{(1-x)^2} < 0 \text{ for } x \text{ in } [0,1)$$

f <u>concave</u> $\Rightarrow$ $x = 0.2$ is <u>max</u> for f

① Integration

Defn. Let $f(x)$ be a given (cont.) fn.

An antiderivative $F(x)$ of $f(x)$ is a fn. such that
$$F'(x) = f(x)$$

Ex: $f(x) = 12x^2 - 6x$

$F(x) = 4x^3 - 3x^2 + C$ is an antiderivative of $f(x)$ for a concrete value of C

and the general antiderivative of $f(x)$ when C = all constants.

Indefinite integral:

$$\int f(x)\,dx = \left\{ \begin{array}{l} \text{general antiderivative} \\ \text{of } f(x) \text{ w.r.t.} \\ \text{the variable } x. \end{array} \right\} = \underbrace{\text{---}}_{\text{one antiderivative of } f(x)} + C$$

Integration rules:

i) $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C, \; n \neq -1$

ii) $\int \frac{1}{x}\,dx = \ln|x| + C$

iii) $\int u \pm v\,dx = \int u\,dx \pm \int v\,dx$

$\int c \cdot u\,dx = c \cdot \int u\,dx$

u, v: expr. in x $\quad c$: const.

$\left(\dfrac{x^{n+1}}{n+1}\right)' = \dfrac{1}{\cancel{n+1}} \cancel{(n+1)} \cdot x^n = x^n$

$x > 0: (\ln x)' = 1/x$

$x < 0: (\ln(-x))' = \frac{1}{-x} \cdot (-1) = 1/x$

iv) $\int e^x \, dx = e^x + C$

<u>Ex:</u> $\int x^2 - x + 1 \, dx = \underline{\underline{\frac{x^3}{3} - \frac{x^2}{2} + x + C}}$

$$\int \frac{1}{x^2} \, dx = \int x^{-2} \, dx = \frac{x^{-1}}{-1} + C = \underline{\underline{-\frac{1}{x} + C}}$$

$$\int \sqrt{x} \, dx = \int x^{1/2} \, dx = \frac{x^{3/2}}{3/2} + C = \underline{\underline{\frac{2}{3} x\sqrt{x} + C}}$$

② <u>Integration techniques:</u>

<u>Ex:</u> $\int x \cdot e^x \, dx$ $\qquad \int x \cdot \ln x \, dx$

(a) <u>Integration by parts</u>

$$\boxed{\int u'v \, dx = uv - \int uv' \, dx}$$

<u>Ex:</u> $\cancel{\int \underbrace{x}_{u'} \cdot \underbrace{e^x}_{v} \, dx = \frac{x^2}{2} \cdot e^x - \int \frac{x^2}{2} e^x \, dx}$

$\cancel{u = x^2/2 \qquad v = e^x}$

$\cancel{u' = x \qquad v' = e^x}$

$$\int \underbrace{x}_{v} \underbrace{e^x}_{u'} \, dx = xe^x - \int e^x \cdot 1 \, dx$$

$u = e^x \qquad v = x$

$u' = e^x \qquad v' = 1$

$$= xe^x - \int e^x dx$$

$$= \underline{\underline{xe^x - e^x + C}}$$

"inverted product rule"

= method to integrate prod.

$$(uv)' = u'v + uv'$$

$$uv = \int u'v + uv' \, dx$$

$$= \int \underline{u'v \, dx} + \int uv' \, dx$$

$$\Downarrow$$

$$\int u'v \, dx = uv - \int uv' \, dx$$

Ex: $\int x \cdot \ln x \, dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x} \, dx$

$$\boxed{\begin{array}{ll} u = \frac{1}{2}x^2 & v = \ln x \\ u' = x & v' = 1/x \end{array}}$$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x \, dx$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\frac{x^2}{2} + C$

$= \underline{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}$

Ex: $\int \ln x \, dx = \int 1 \cdot \ln x \, dx = x \cdot \ln x - \int \cancel{1 \cdot \ln x}\; x \cdot \frac{1}{x} \, dx$

$$\boxed{\begin{array}{ll} u = x & v = \ln x \\ u' = 1 & v' = 1/x \end{array}}$$

$= x \ln x - \int 1 \, dx = \underline{\underline{x \ln x - x + C}}$

Ex: $\int x \cdot \sqrt{x^2+1} \, dx$ maybe not integration by parts

(b) Substitution "chain rule inverted"

Ex: $\int \frac{2}{1-x} \, dx = \int \frac{2}{u} \frac{du}{-1} = \int \frac{-2}{u} \, du$

$u = 1-x$

General formula for substitution $\rightarrow$ $\boxed{du = u' \cdot dx}$

$du = -1 \, dx$

$dx = \frac{du}{-1}$

$= -2 \cdot \int \frac{1}{u} \, du = -2 \ln|u| + C$

$= \underline{\underline{-2 \ln|1-x| + C}}$

$(-2 \ln|1-x| + C)' = -2 \cdot \frac{1}{1-x} \cdot (-1) = \frac{2}{1-x}$

Ex: $\int x\cdot\sqrt{x^2+1}\,dx = \int x\cdot\sqrt{u}\cdot\frac{du}{2x} = \int \frac{\cancel{x}\cdot\sqrt{u}}{2\cancel{x}}\,du$

$\boxed{\begin{array}{c} u = x^2+1 \\ du = 2x\,dx \end{array}} \Rightarrow dx = \frac{du}{2x}$

$= \int \frac{1}{2}u^{1/2}\,du = \frac{1}{2}\cdot\frac{u^{3/2}}{3/2} + C = \frac{1\cdot u^{3/2}}{3} + C$

$= \frac{1}{3}(x^2+1)^{3/2} + C = \underline{\underline{\frac{1}{3}(x^2+1)\cdot\sqrt{x^2+1} + C}}$

Ex: $\int \frac{2e^x}{e^x+e^{-x}}\,dx = \int \frac{2u}{u+1/u}\,\frac{du}{e^x} = \int \frac{2u}{(u+1/u)u}\,du$

$\boxed{\begin{array}{c} u = e^x \\ du = e^x\,dx \end{array}}$ $dx = \frac{du}{e^x}$

$= \int \frac{2u}{u^2+1}\,du = \int \frac{\cancel{2u}}{v}\,\frac{dv}{\cancel{2u}} = \int \frac{1}{v}\,dv$

$\boxed{\begin{array}{c} v = u^2+1 \\ dv = 2u\cdot du \end{array}}$

$= \ln|v| + C = \ln(u^2+1) + C = \underline{\underline{\ln(e^{2x}+1) + C}}$

(c) Integration of fractions

$$\int \frac{2}{1-x}\,dx = \int \frac{2}{u}\,\frac{du}{-1} = \frac{2}{-1}\ln|1-x| + C = \underline{\underline{-2\ln|1-x|+C}}$$

$$\boxed{\begin{aligned} u &= 1-x \\ du &= -1\,dx \end{aligned}}$$

$$\boxed{\int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln|ax+b| + C}$$

Ex: $\int \frac{x^2}{1-x}\,dx$

$$= \int -x-1+\frac{1}{1-x}\,dx$$

$$= -\frac{1}{2}x^2 - x + \frac{1}{-1}\ln|1-x| + C$$

$$= \underline{\underline{-\frac{1}{2}x^2 - x - \ln|1-x| + C}}$$

Method: polynomial division

$$x^2 : -x+1 = \underline{-x-1}$$

quot.

$$\underline{-(x^2-x)}$$

$$x$$

$$\underline{-(x-1)}$$

1 = remainder

$$\frac{x^2}{1-x} = -x-1+\frac{1}{1-x}$$

Ex: $\int \frac{2}{1-x^2}\,dx = \int \frac{2}{u}\,\frac{du}{-2x} = -\int \frac{1}{x\cdot u}\,du$

$$\boxed{\begin{aligned} u &= 1-x^2 \\ du &= -2x\,dx \end{aligned}}$$

$$x^2 = 1-u$$

$$x = \pm\sqrt{1-u}$$

Partial fractions: $1-x^2 = (1-x)(1+x)$

$$\frac{2}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} \quad \Big| \cdot (1+x)(1-x)$$

$$2 = A(1+x) + B(1-x)$$

$$2 = A + Ax + B - Bx$$

$$= \underbrace{Ax - Bx} + A + B$$

$$2 = (A-B)x + (A+B)$$

$$\cancel{2 = (A-B)x + A+B}$$

$$\cancel{2 - A - B = (A-B)x}$$

$$\cancel{x = \frac{2-A-B}{A-B}}$$

$$0 = A - B$$

$$2 = A + B$$

$$2 = 2A \qquad \underline{A=1} \quad \underline{B=1}$$

Concl: $\frac{2}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+x}$

$$\int \frac{2}{1-x^2}\,dx = \int \frac{1}{1-x} + \frac{1}{1+x}\,dx = -\ln|1-x| + \ln|1+x| + C$$

$$= \underline{\underline{\ln\frac{|1+x|}{|1-x|} + C}}$$