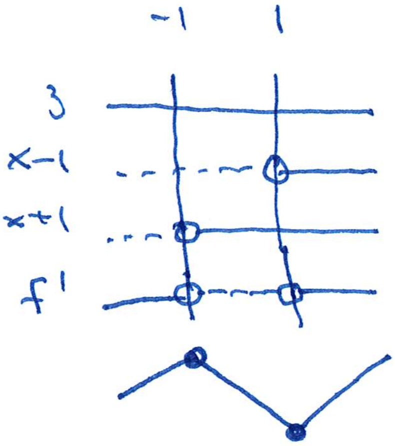
Plan
1 Functions and derivatives
2 Exponential functions and logarithms
3 Higher derivatives and convexity

no time, this will be covered in Lecture 4

① Functions and derivatives



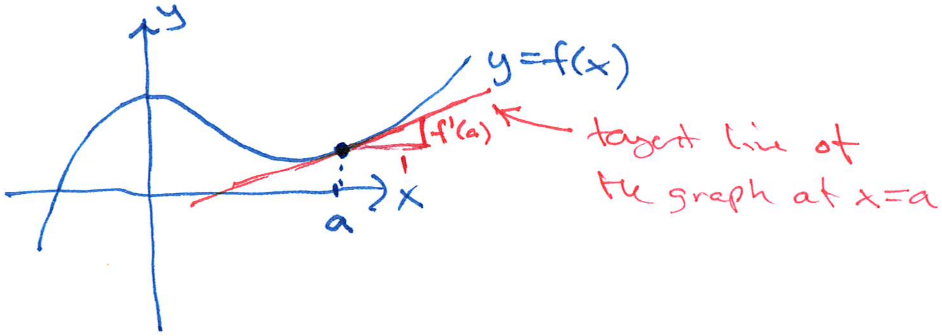
Ex: $f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3 = 3(x^2-1)$

$= 3(x-1)(x+1)$

Interpretation of derivative:

$f'(a)$ = slope of the tangent line of the graph of f at $x=a$



$f'(a) > 0$: function f is increasing close to $x=a$

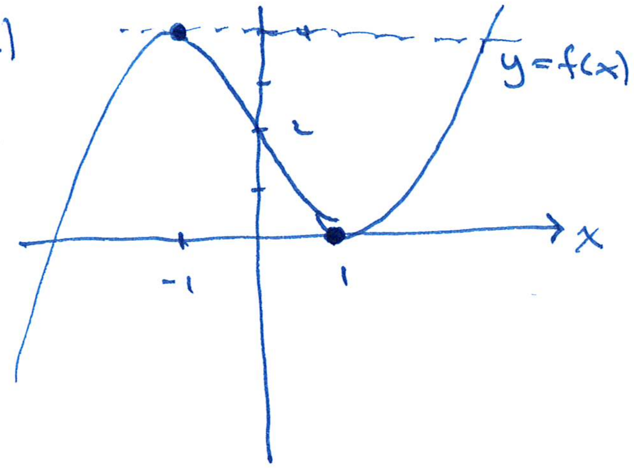
$f'(a) < 0$: — " — decreasing — " —

$f'(a) = 0$: stationary point of f, horisontal tangent line

Ex (cont'd)

$f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3$



when $x \to \infty$:

$f \to \infty$

no max

when $x \to -\infty$

$f \to -\infty$

no min

Max/min

Defn $x=a$ is a max. pt of f if $f(a) \geq f(x)$ for all x

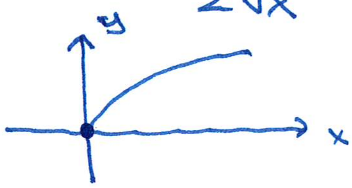
$x=a$ is a min pt of f if $f(a) \leq f(x)$ for all x

Fact: $\boxed{\text{If } x=a \text{ is a max/min for } f \text{ and } (*), \text{ then } f'(a)=0 \text{ } (x=a \text{ is a stationary pt}).}$

Except: $(*)$: If $f'(a)$ exists and $x=a$ is not a boundary pt of D_f.

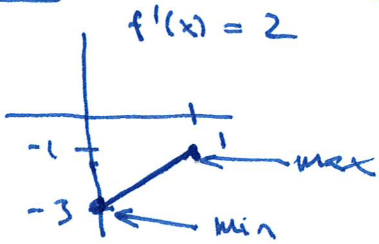
Ex: $f(x) = \sqrt{x}$, $x \geq 0$

$f'(x) = \dfrac{1}{2\sqrt{x}}$ $f'(0)$ does not exist



Ex: $f(x) = 2x-3$, $0 \leq x \leq 1$

$f'(x) = 2$



Defn $x=a$ is a <u>local max</u> of f if $f(a) \geq f(x)$ for all x close to $x=a$

$x=a$ is a <u>local min</u> of f if $f(a) \leq f(x)$ for all x close to $x=a$

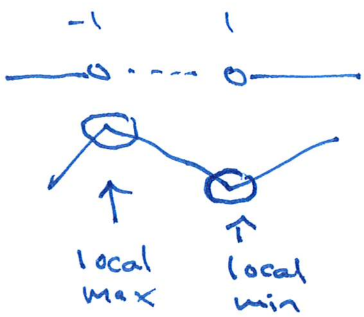
Fact: If $f'(a)=0$ and $f'<0$ to the right, $f'>0$ –"– left

then $x=a$ is a <u>local max</u>

If $f'(a)=0$ and $f'>0$ to the right, $f'<0$ to the left

then $x=a$ is a <u>local min</u>

Ex: $f'(x) = 3(x-1)(x+1)$



Ex: $f(x) = x^4 - 4x^2$

$x \to -\infty$: $f \to \infty$ $x \to \infty$: $f \to \infty$

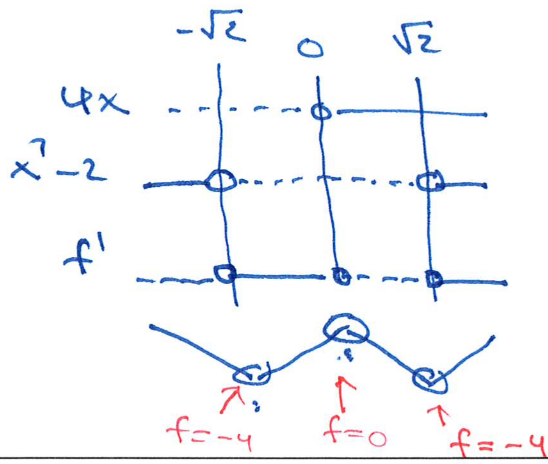
$f'(x) = 4x^3 - 8x$

$= 4x(x^2-2)$

$f'(x)=0$: $4x(x^2-2) = 0$

$x=0$ or $x^2=2$

$x = \pm\sqrt{2}$



Concl: no max

$x = \pm\sqrt{2}$ min

$f_{min} = -4$

Local max: $x=0$

Local min: $x = \pm\sqrt{2}$

Derivation rules:

i) $(x^n)' = n x^{n-1}$ for all n

ii) $(u \pm v)' = u' \pm v'$
$(c \cdot u)' = c \cdot u'$
u, v: expr in x
c: const.

iii) $(u \cdot v)' = u' \cdot v + u \cdot v'$

iv) $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

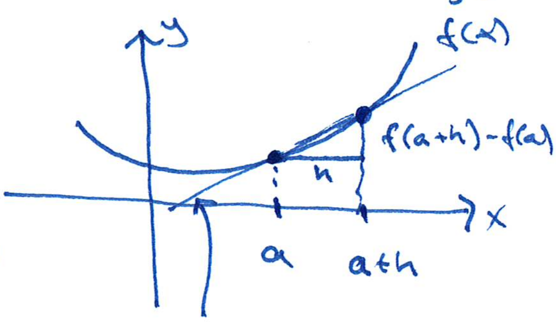
u, v: expr. in x

v) $(e^x)' = e^x$ $\quad (a^x)' = a^x \cdot \ln(a)$

$(\ln x)' = \frac{1}{x}$ $\quad (\log_a x)' = \frac{1}{x \cdot \ln(a)}$

Defn:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



Secant, slope:

$$\frac{f(a+h) - f(a)}{h}$$

vi) Chain rule: $f(x) = h(u(x))$

$\Rightarrow f'(x) = h'(u(x)) \cdot u'(x)$

$x \xrightarrow{u} \underset{u}{\underset{\shortparallel}{u(x)}} \xrightarrow{h} \underset{h(u(x))}{\underset{\shortparallel}{h(u)}}$

Ex: $f(x) = x^5 - 4x^3 + 3x^2 - 7$

$f'(x) = 5x^4 - 4 \cdot 3x^2 + 3 \cdot 2x = \underline{5x^4 - 12x^2 + 6x}$

$f(x) = \frac{x-1}{x+1} = \frac{u}{v}$ $\quad u = x-1$, $v = x+1$

$f'(x) = \frac{1 \cdot (x+1) - (x-1) \cdot 1}{(x+1)^2} = \underline{\frac{2}{(x+1)^2}}$

$f(x) = \underbrace{x}_{u} \cdot \underbrace{e^x}_{v}$ $\quad f'(x) = 1 \cdot e^x + x \cdot e^x$
$= \underline{(x+1)e^x}$

Ex: $f(x) = \sqrt{x^2+1}$

$x \xrightarrow{u} x^2+1 = u \xrightarrow{h} \sqrt{u}$

$f'(x) = h'(u) \cdot u'(x)$

$u(x) = x^2+1$ $\qquad h(u) = \sqrt{u} = u^{1/2}$

$u' = 2x$ $\qquad h' = \frac{1}{2} u^{-1/2}$

$= \frac{1}{2\sqrt{u}} \cdot 2x$ $\qquad = \frac{1}{2} \cdot \frac{1}{u^{1/2}}$

$\qquad = \frac{1}{2\sqrt{u}}$

$= \frac{1}{\cancel{2}\sqrt{x^2+1}} \cdot \cancel{2}x = \underline{\underline{\frac{x}{\sqrt{x^2+1}}}}$

$f(x) = \ln(x-1)$

$x \xrightarrow{u} x-1 \xrightarrow{h} \ln(u)$, $\quad x-1 = u$

$u(x) = x-1 \qquad h(u) = \ln(u)$

$u' = 1 \qquad h' = \frac{1}{u}$

$f'(x) = \frac{1}{u} \cdot 1$

$= \underline{\underline{\frac{1}{x-1}}}$

$f(x) = \ln(1-x)$

$f'(x) = \frac{1}{u} \cdot (-1)$

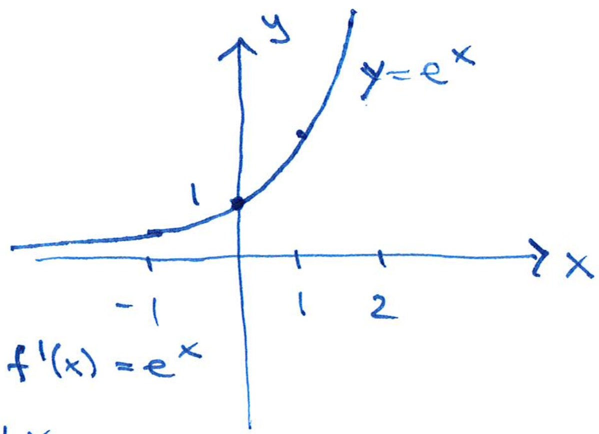
$= \underline{\underline{-\frac{1}{1-x}}}$

(2) Exponential fn. and logarithms

Base: $a > 0$

$f(x) = e^x$

$f'(x) = e^x$



Very useful:

Base: $a = e = 2.718\ldots$ Euler's number

$f(x) = e^x$: $f'(x) = e^x$

- defined for all x
- positive —"—
- increasing
- limits: $e^x \to \infty \ (x \to \infty)$
  $e^x \to 0 \ (x \to -\infty)$

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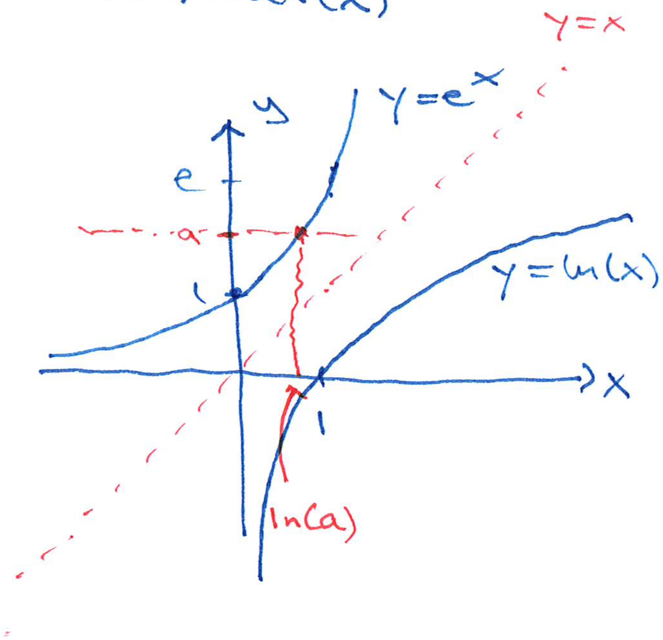
$f(x) = \ln(x)$ natural logarithm (base e), inverse fn. of e^x



$\ln(a) = b \iff e^b = a$

Note: $f(x) = \ln(x)$
- defined for $x > 0$
- increasing $(\ln x)' = \frac{1}{x}$
- limits: $\ln x \to \infty \quad (x \to \infty)$
  $\ln x \to -\infty \quad (x \to 0^+)$

Useful rules for logarithms:

i) $\ln(ab) = \ln(a) + \ln(b)$

ii) $\ln(a/b) = \ln(a) - \ln(b)$

iii) $\ln(a^n) = n \cdot \ln(a)$

for exponentials:

$e^a \cdot e^b = e^{a+b}$

$e^a / e^b = e^{a-b}$

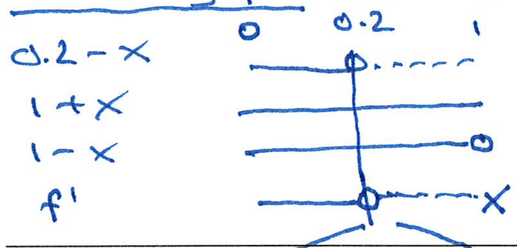
$(e^a)^n = e^{a \cdot n}$

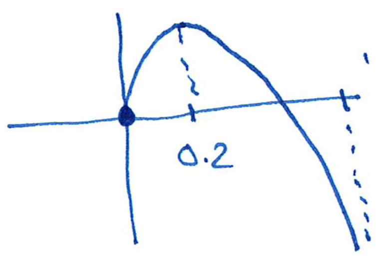
Ex: $f(x) = \frac{3}{5}\ln(1+x) + \frac{2}{5}\ln(1-x) \quad 0 \leq x < 1$

$$f'(x) = 0.6 \cdot \frac{1}{1+x} \cdot 1 + 0.4 \cdot \frac{1}{1-x} \cdot (-1)$$

$$= \frac{0.6}{1+x} - \frac{0.4}{1-x} = \frac{0.6 \cdot (1-x) - 0.4(1+x)}{(1+x)(1-x)} = \frac{0.2 - x}{(1+x)(1-x)}$$

Stationary pts: $x = 0.2$





$x \to 1$:

$f(x) \to -\infty$

max: $x = 0.2$