Plan

1 Vectors and matrices
2 Matrix multiplication
3 Inverse matrices and determinants

(1) Vectors and matrices

Defn: An $m \times n$-matrix is a rectangular array of numbers with m rows and n col's.

entry in A, row 1, col 2.

$$A = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{n} \Bigg\} m$$

Ex: $A = \begin{pmatrix} 1 & 2 & 7 \\ 4 & -1 & 0 \end{pmatrix}$ 2×3-matrix

Defn: An n-vector is an $n \times 1$-matrix.

$$\vec{v} = \underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$ column vectors

Ex: $\underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ 2-vector

Operations:

i) Addition: $A+B$, $\underline{v}+\underline{w}$

- possible when they have the same size
- computed position by position

Ex: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & -1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 3 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 4 \\ 7 & 1 & 4 \end{pmatrix}$

$\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

ii) Scalar multiplication: $c \cdot A$, $c \cdot \underline{v}$

- computed position by position

scalar = number

Ex: $3 \cdot \begin{pmatrix} 0 & 1 & 1 \\ 3 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 3 \\ 9 & 6 & 6 \end{pmatrix}$

$3 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$

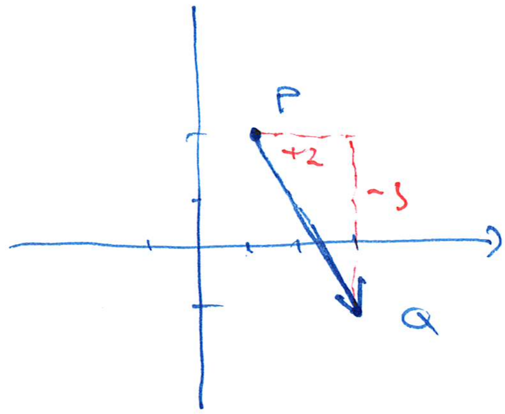
Ex: $3 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 7 \end{pmatrix} = \underbrace{\begin{pmatrix} 6 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 7 \end{pmatrix}} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

$\begin{pmatrix} 6 \\ 3 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} + \begin{pmatrix} -4 \\ -7 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

Vectors:

$P = (1,2)$

$Q = (3,-1)$



$$\underline{v} = \overrightarrow{PQ} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 3-1 \\ -1-2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\|\underline{v}\| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$$

length of the vector $\underline{v}$

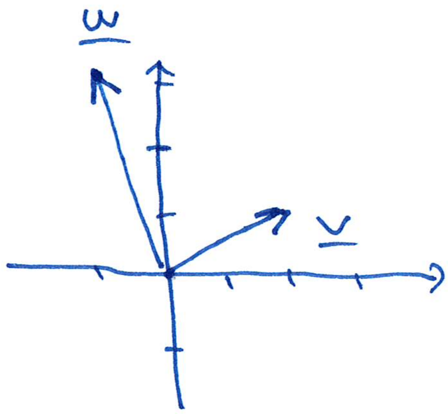
In general:

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} : \quad \|\underline{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

Inner product:

Ex: $\underline{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\underline{w} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$



$$\underline{v} \cdot \underline{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$= 2 \cdot (-1) + 1 \cdot 3 = 1$$

In general: The inner product is defined when the vectors have the same size.

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad \underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} : \quad \underline{v} \cdot \underline{w} = v_1 w_1 + v_2 \cdot w_2 + \dots + v_n \cdot w_n$$

Fact: $\underline{v} \cdot \underline{w} = 0 \iff \underline{v} \perp \underline{w}$ (the angles between the vectors is 90°)

$\underline{v} \cdot \underline{w} > 0$: angle $< 90^\circ$

$\underline{v} \cdot \underline{w} < 0$: angle $> 90^\circ$

(2) Matrix multiplication

Ex: $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1\cdot 3 + 2 \cdot 1 \\ 2 \cdot 3 + 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$

$2\times 2 \quad = \quad 2\times 1 \qquad\qquad 2 \times 1$

Fact: $\underset{m\times n}{A} \cdot \underset{n \times p}{B}$ is defined when # cols(A) = # rows(B) and the product is an $m\times p$-matrix.

Ex: $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 5 \\ 2 & 1 \end{pmatrix}$

$2\times \underbrace{2 \quad\quad 2}_{=} \times 2$

$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 2 & 7 \end{pmatrix}$

$2\times \underbrace{2 \quad\quad 2}_{=} \times 2$

Note: $AB \neq BA$

Ex: $x+y+z+w=4$

$x-y+z-w=2$

$x+2y \quad -4w=-1$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 2 & 0 & -4 \end{pmatrix} \cdot \quad \underline{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$$

coeff. matrix

$$\underbrace{3 \times 4 \qquad 4 \times 1}_{=}$$

$$= \begin{pmatrix} x+y+z+w \\ x-y+z-w \\ x+2y \quad -4w \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} = \underline{b}$$

Linear system in matrix form: $\boxed{A \cdot \underline{x} = \underline{b}}$

Compare: $ax = b \underset{a \neq 0}{\Rightarrow} x = \frac{1}{a} \cdot b = b/a$

$2x = 8 \qquad x = 8/2 = 4$

3 Inverse matrices and determinants

Defn: The $n \times n$ identity matrix is $I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Property: $\left\{ \begin{matrix} A \cdot I = A \\ \text{and} \\ I \cdot A = A \end{matrix} \right\}$ for any matrix A

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = A$

Inverse Matrix:

A $n \times n$-matrix

Defn: An inverse matrix of A is a matrix B such that $\left\{ \begin{matrix} A \cdot B = I \\ \text{and} \\ B \cdot A = I \end{matrix} \right\}$

Fact:
- not all matrices have an inverse
- if it has an inverse, then the inverse is unique and is written A^{-1}.

Ex: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

2×2-matrix

i) $\det(A) = |A| = ad - bc$

Fact:

A has an inverse $\Leftrightarrow$ $|A| \neq 0$

ii) If $|A| \neq 0$, then

$$A^{-1} = \frac{1}{|A|} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$$\det(A) = |A| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 2 \cdot 2 - 1 \cdot 1 = 3 \neq 0$$

$\Rightarrow$ A has an inverse A^{-1}

$$A^{-1} = \underline{\frac{1}{3} \cdot \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}} = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}$$

$$A \cdot A^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \frac{1}{3} \cdot \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \frac{1}{3} \underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}$$

$$= \frac{1}{3} \cdot \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Ex: $2x + y = 17$
$x + 2y = -13$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 17 \\ -13 \end{pmatrix}$$
$$\underset{A}{\shortparallel} \quad \underset{\underline{x}}{} \quad \underset{\underline{b}}{\shortparallel}$$

$A\underline{x} = \underline{b} \quad | \; A^{-1}\cdot$

We know:
$$A^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$A^{-1} \cdot A\underline{x} = A^{-1}\underline{b}$$
$$I\underline{x} = A^{-1}\underline{b}$$
$$\underline{x} = A^{-1}\underline{b}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 17 \\ -13 \end{pmatrix}$$
$$= \frac{1}{3}\begin{pmatrix} 47 \\ -43 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 47/3 \\ -43/3 \end{pmatrix}}}$$

Summary:

If A is an $m \times n$-matrix $(m \neq n)$, then A does not have an inverse.

If A is an $n \times n$-matrix (square matrix), then we have:

i) A has an inverse $\Leftrightarrow \det(A) \neq 0$
(A invertible)

ii) If $|A| \neq 0$, $A^{-1} = \frac{1}{|A|} \cdot \operatorname{adj}(A)$

n=2: $|A| = ad - bc$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \operatorname{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Powers of matrices: A^n can be computed when A is square

Ex: $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$1 \cdot 1 + 2 \cdot 2$ ''

$$A^2 = A \cdot A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

$$A^3 = A \cdot \underbrace{A \cdot A}_{A^2} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 13 & 14 \\ 14 & 13 \end{pmatrix}$$

$$\vdots$$

$$A^n = ?$$

Diagonal matrix: $D = \begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \; d_n \end{pmatrix}$

Ex: $\begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix}^2 = \begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 7^2 & 0 \\ 0 & 2^2 \end{pmatrix}$

$$\begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 7^n & 0 \\ 0 & 2^n \end{pmatrix}$$

General method to compute A^{-1}

Method

A $n \times n$-matrix

i) Form a matrix $(A|I)$

ii) Transform $(A|I) \to \dots \to (B|C)$ into a reduced echelon form using elementary row operations.

iii) Conclude:

If $B = I$ then $A^{-1} = C$

If $B \neq I$ then A is not invertible

Reduced echelon form:

Matrix in echelon form and satisfy the additional properties:

i) each pivot $= 1$

ii) all entries over a pivot is zero

Alt:

$$A^{-1} = \frac{1}{-3}\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix}$$

Ex:

$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right) \begin{matrix} \\ -2 \end{matrix} \to \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -3 & -2 & 1 \end{array}\right) \cdot (-1/3)$$

echelon form

$$\to \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 2/3 & -1/3 \end{array}\right) \begin{matrix} -2 \\ \end{matrix} \to \left(\begin{array}{cc|cc} 1 & 0 & -1/3 & 2/3 \\ 0 & 1 & 2/3 & -1/3 \end{array}\right)$$

Since $B = I$, $A^{-1} = C = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix}$