Problems

Problem 3.1 Let A, B, and C be the 3×3 matrices

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Compute the following matrix expressions:

a) AB b) BA c) A^2

d) B^2 e) $(A+B)^2$ f) ABC

Problem 3.2 Give an example of a symmetric and a nonsymmetric 4×4 matrix.

Problem 3.3 Let A, B, and C be any $n \times n$ matrices. Simplify the following matrix expressions:

a) $AB(BC - CB) + (CA - AB)BC + CA(A - B)C$

b) $(A - B)(C - A) + (C - B)(A - C) + (C - A)^2$

Problem 3.4 We consider a linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 3 & 1 & 5 \\ 5 & -3 & 2 \\ 4 & -3 & -1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix}$$

a) Write out the linear system of equations.

b) Determine whether A is invertible, and find A^{-1} if it exists.

c) How many solutions does the linear system have?

Problem 3.5 Compute the matrix $A^T A$ when A is the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 6 & 5 \end{pmatrix}$$

Problem 3.6 Compute $|A|$ using cofactor expansion along the first column, and then along the third row. Compare both the results and the computations.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{pmatrix}$$

Problem 3.7 Compute the following determinants using elementary row operations:

a) $\begin{vmatrix} 3 & 1 & 5 \\ 9 & 3 & 15 \\ -3 & -1 & -5 \end{vmatrix}$ b) $\begin{vmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{vmatrix}$ c) $\begin{vmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{vmatrix}$

Problem 3.8 Use determinants and cofactors to determine when A is invertible, and to compute A^{-1} when possible:

a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{pmatrix}$ b) $A = \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Problem 3.9 Use determinants to check whether the following vectors are linearly independent:

a) $\mathbf{v}_1 = (3, 5), \quad \mathbf{v}_2 = (7, -1)$

b) $\mathbf{v}_1 = (1, 1, 1), \quad \mathbf{v}_2 = (1, 2, -1), \quad \mathbf{v}_3 = (1, 4, 1)$

c) $\mathbf{v}_1 = (1, 1, 1, 1), \ \mathbf{v}_2 = (1, 2, -2, -4), \ \mathbf{v}_3 = (0, 1, 4, 1), \ \mathbf{v}_4 = (1, 0, 0, -1)$

Problem 3.10 Let A and B be 3×3-matrices with $|A| = 2$ and $|B| = -5$. Find $|AB|$, $|-3A|$, $|A^{-1}|$, and $|-2A^T|$.

Problem 3.11 Compute all 3-minors of A, and use this to find $\text{rk}(A)$. How many 2-minors are there?

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 1 & -2 & -2 & 1 \end{pmatrix}$$

Problem 3.12 Determine the rank of A for all values of the parameters s and t:

a) $A = \begin{pmatrix} 1 & s & s^2 \\ 1 & 0 & 1 \\ s & -1 & s \end{pmatrix}$ b) $A = \begin{pmatrix} t+6 & 5 & 6 \\ -1 & t & -6 \\ 1 & 1 & t+7 \end{pmatrix}$

Problem 3.13 Let A be any $m \times n$-matrix, and consider the matrix $A^T A$. Show that:

a) $\text{Null}(A^T A) = \text{Null}(A)$ b) $\text{rk}(A^T A) = \text{rk}\, A$