It is the special case where $b_1 = b_2 = \ldots = b_m = 0$. If we write down the augmented matrix of a homogeneous linear system, the last column consists of zeros, and it is easy to see that this will not change in a Gaussian process. Therefore, there is never a pivot position in the last column of a homogeneous linear system, and this means that it is always consistent. In fact, the last column is often omitted when solving homogeneous systems.

Note that $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)$ is a solution of any homogeneous linear system. It is called the *trivial solution*. There are only two possibilities: Either the trivial solution is the only solution, or there are other solutions, called *nontrivial solutions*, in addition to the trivial one. In the first case, the system has one solution, and in the second case, the system must have infinitely many solutions. It follows from Theorem 1.6 that we can find the number of solutions using rank:

Proposition 1.7. An $m \times n$ homogeneous linear system with coefficient matrix A has only the trivial solution if $\mathrm{rk}(A) = n$, and it has nontrivial solutions if $\mathrm{rk}(A) < n$.

Problems

Problem 1.1 Write down the coefficient matrix and the augmented matrix of the following linear systems:

a)
$$\begin{aligned} 7x + 2y &= 4 \\ 4x - 4y &= 7 \end{aligned}$$

b)
$$\begin{aligned} y - z + w &= 1 \\ x + y + z - 2w &= 3 \\ x - 4y + 7z + 3w &= 14 \end{aligned}$$

Problem 1.2 Write down linear systems with the following augmented matrices:

a) $\left(\begin{array}{ccc|c} 2 & 1 & 0 & 7 \\ 1 & 3 & 4 & 5 \\ 5 & -4 & 2 & 13 \end{array}\right)$

b) $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 1 & 2 & 4 & 12 \\ 1 & 3 & 9 & 19 \end{array}\right)$

c) $\left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 12 \\ 2 & -3 & 4 & 7 & 10 \end{array}\right)$

Problem 1.3 Use substitution to solve the following linear system, and then solve the same linear system using Gaussian elimination:

$$\begin{aligned} x + y + z &= 7 \\ x + 2y + 4z &= 12 \\ x + 3y + 9z &= 19 \end{aligned}$$

Problem 1.4 Solve the following linear systems by Gaussian elimination. Use the pivot positions to determine the number of solutions:

a)
$$\begin{aligned} -4x + 6y + 4z &= 4 \\ 2x - y + z &= 1 \end{aligned}$$

b)
$$\begin{aligned} 6x + y &= 7 \\ 3x + y &= 4 \\ -6x - 2y &= 1 \end{aligned}$$

Problem 1.5 Solve the following linear system, and give a geometric description of its solutions:

a) $3x - 4y = 6$

b)
$$\begin{aligned} x + y + z &= 4 \\ 2x - y + 3z &= 3 \\ 3x \quad\quad + 4z &= 7 \end{aligned}$$

c)
$$\begin{aligned} 2x + y - 4z &= 3 \\ 3x - 2y + z &= 1 \end{aligned}$$

Problem 1.6 Determine all values of h such that the following linear system has solutions, and describe the solutions geometrically in these cases:

a) $hx - hy = 3$

b)
$$\begin{aligned} x + 2y &= h \\ 3x + 6y &= 12 \end{aligned}$$

c)
$$\begin{aligned} x + y + z &= 1 \\ 2x - y + 3z &= 4 \\ x + 7y - z &= h \end{aligned}$$