Problems for Lecture 4

1. Compute $f'(x)$ and $f''(x)$.

a) $f(x) = x^4 - x\sqrt{x} + 3x \quad , \quad x > 0$

b) $f(x) = x\sqrt{x^2+1}$

c) $f(x) = \dfrac{x+1}{x^2-3x+2} \quad , \quad x \neq 1, 2$

d) $f(x) = xe^x - x^2e^{-x} + e^{2x-1}$

e) $f(x) = \ln(x) - \ln(x-1) \quad , \quad x > 1$

f) $f(x) = \ln(x^3 - x^2) \quad , \quad x > 1$

2. Is f convex, concave, or neither?

a) $f(x) = x^2$

b) $f(x) = x^a \quad (a > 0)$

c) $f(x) = e^x$

d) $f(x) = \ln(x) \quad , \quad x > 0$

e) $f(x) = \ln(x^3 - x^2) \quad , \quad x > 1$

Solutions for Lecture 4

1.

a) $f'(x) = 4x^3 - \frac{3}{2}\sqrt{x} + 3$

$f''(x) = 12x^2 - \frac{3}{4\sqrt{x}}$

b) $f'(x) = 1 \cdot \sqrt{x^2+1} + x \cdot \frac{1}{\cancel{2}\sqrt{x^2+1}} \cdot \cancel{2}x = \frac{\sqrt{x^2+1}\cdot\sqrt{x^2+1} + x^2}{\sqrt{x^2+1}}$

$= \frac{2x^2+1}{\sqrt{x^2+1}}$

$f''(x) = \frac{4x \cdot \sqrt{x^2+1} - (2x^2+1) \cdot \frac{1}{\cancel{2}\sqrt{x^2+1}} \cdot \cancel{2}x}{x^2+1}$

$= \frac{4x(x^2+1) - x(2x^2+1)}{(x^2+1)\sqrt{x^2+1}}$

$= \frac{2x^3+3x}{(x^2+1)^{3/2}}$

c) $f'(x) = \left(\frac{x+1}{x^2-3x+2}\right)' = \frac{1 \cdot (x^2-3x+2) - (x+1)\cdot(2x-3)}{(x^2-3x+2)^2}$

$= \frac{(x^2-3x+2) - (2x^2-x-3)}{(x^2-3x+2)^2} = \frac{-x^2-2x+5}{(x^2-3x+2)^2}$

$f''(x) = \frac{(-2x-2)(x^2-3x+2)^2 - (-x^2-2x+5)\cdot 2(x^2-3x+2)\cdot(2x-3)}{(x^2-3x+2)^4}$

$= \frac{(-2x-2)(x^2-3x+2) - (4x-6)(-x^2-2x+5)}{(x^2-3x+2)^3} = \frac{2x^3+6x^2-30x+34}{(x^2-3x+2)^3}$

d) $f'(x) = 1\cdot e^x + \cancel{x}e^x - 2xe^{-x} - x^2 e^{-x}\cdot(-1) + e^{2x-1}\cdot 2$

$= (x+1)e^x + (x^2-2x)e^{-x} + 2e^{2x-1}$

$f''(x) = (x+2)e^x + (2x-2)e^{-x} + (x^2-2x)e^{-x}\cdot(-1) + 4e^{2x-1}$

$= (x+2)e^x + (-x^2+4x-2)e^{-x} + 4e^{2x-1}$

e) $f(x) = \ln(x) - \ln(x-1)$

$f'(x) = \frac{1}{x} - \frac{1}{x-1} = \frac{x-1-x}{x(x-1)} = \frac{-1}{x(x-1)}$

$f''(x) = -\frac{1}{x^2} - \left(-\frac{1}{(x-1)^2}\right)\cdot(+1) = -\frac{1}{x^2} + \frac{1}{(x-1)^2} = \frac{(x-1)^2 - x^2}{x^2(x-1)^2}$

$= \frac{1-2x}{x^2(x-1)^2}$

f) $f'(x) = \frac{1}{x^3-x^2}\cdot(3x^2-2x) = \frac{3x^2-2x}{x^2(x-1)} = \frac{x(3x-2)}{x^2(x-1)} = \frac{3x-2}{x(x-1)}$

$f''(x) = \left(\frac{3x-2}{x^2-x}\right)' = \frac{3(x^2-x)-(3x-2)\cdot(2x-1)}{(x^2-x)^2} = \frac{-3x^2+4x-2}{x^2(x-1)^2}$

2. a) $f'(x) = 2x$ $\quad f''(x) = 2 > 0$ for all x $\rightarrow$ convex

b) $f'(x) = ax^{a-1}$ $\quad f''(x) = a\cdot(a-1)x^{a-2}$

since $a>0$, $x^{a-2}>0$,
we have that

$f''(x) > 0$ when $a>1$ $\rightarrow$ $a>1$: f convex
$f''(x) < 0$ " $a<1$ $\quad$ $0<a<1$: f concave

c) $f' = f'' = e^x > 0$ $\rightarrow$ convex

d) $f' = \ln x$ $\quad f'' = \frac{1}{x} = x^{-1}$ $\quad f'' = -\frac{1}{x^2} < 0$ $\rightarrow$ f concave

e) $f'' = \frac{-3x^2+4x-2}{x^2(x-1)^2}$ by (f), and $x^2, (x-1)^2 > 0$.

$-3x^2+4x-2 = 0$

$x = \frac{-4 \pm \sqrt{16-4\cdot 6}}{2\cdot(-3)}$ no real Solution $\rightarrow$ no intersection with x-axis $\rightarrow$ f concave

$f''(x) < 0$ for all $x>1$