Plan:

1. Higher order derivatives and convexity
2. Optimization in one variable
3. Integration ←

Reading: [MK] Ch. 13–14, App. A4

See also [DE], Appendix on Integration (from the web page of GRA6035)

Review: Derivation rules

1. $(x^n)' = nx^{n-1}$
2. $(u \pm v)' = u' \pm v'$ $\quad u, v$ functions in x
3. $(c \cdot u)' = c \cdot u'$ $\quad c$ constant
4. $(u \cdot v)' = u'v + uv'$
5. $(u/v)' = \dfrac{u'v - uv'}{v^2}$
6. $(e^x)' = e^x$ $\quad e = 2{,}71828\ldots$ Euler's number

   $(a^x)' = a^x \cdot \ln(a)$ $\quad a > 0$
7. $(\ln x)' = \dfrac{1}{x}$ $\quad \ln$ = natural logarithm (base e)

   $(\log_a x)' = \dfrac{1}{x} \cdot \dfrac{1}{\ln(a)}$
8. Chain rule: $f = \underbrace{f(u)}_{\text{outer function}}$, where $\underbrace{u = u(x)}_{\text{kernel}}$

$$\frac{df}{dx} = \frac{df}{du} \cdot \underbrace{\frac{du}{dx}}_{u'}$$

Ex: $f(x) = e^{-x^2} = \underline{e^u}$, $u = \underline{-x^2}$

$f'(x) = \underset{\frac{df}{du}}{\underset{\uparrow}{e^u}} \cdot \underset{\frac{du}{dx} = u'(x)}{\underset{\uparrow}{(-2x)}} = e^{-x^2} \cdot (-2x) = \underline{-2x\, e^{-x^2}}$

Ex: $f(x) = e^x + e^{-x}$ $\leftarrow$ e^u , $u = \underline{-x}$

$f'(x) = e^x + e^{-x} \cdot (-1)$

$= \underline{\underline{e^x - e^{-x}}}$

$(e^{-x})' = e^u \cdot (-1)$

$= e^{-x} \cdot (-1)$

Ex: $f(x) = \frac{3}{5} \cdot \underline{\ln(1+x)} + \frac{2}{5} \ln(1-x)$

$f'(x) = \frac{3}{5} \cdot \frac{1}{1+x} + \frac{2}{5} \cdot \frac{1}{1-x} \cdot (-1)$

$= \frac{3}{5} \frac{1}{1+x} - \frac{2}{5} \frac{1}{1-x}$

$= \frac{3(1-x) - 2(1+x)}{5 \cdot (1+x)(1-x)} = \underline{\underline{\frac{1-5x}{5(1+x)(1-x)}}}$

$(\ln(1+x))'$

$(\ln(u)$, $u = \underline{1+x}$

$= \frac{1}{u} \cdot 1 = \frac{1}{u}$

$= \underline{\frac{1}{1+x}}$

Formula:

$$(e^{u(x)})' = e^{u(x)} \cdot u'(x)$$

$$(\ln u(x))' = \frac{1}{u(x)} \cdot u'(x) = \underline{\frac{u'(x)}{u(x)}}$$

Applications:

① Higher order derivatives and convexity

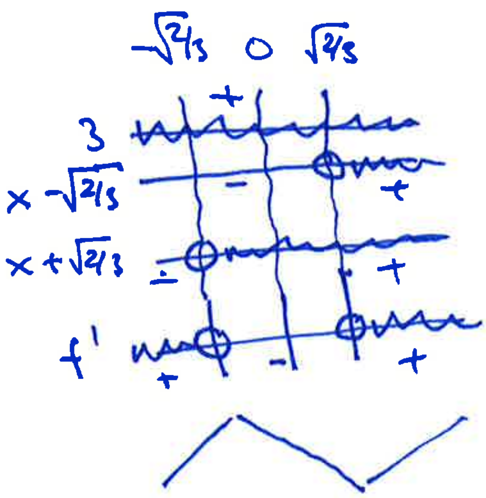
Ex: $f(x) = x^3 - 2x + 1$

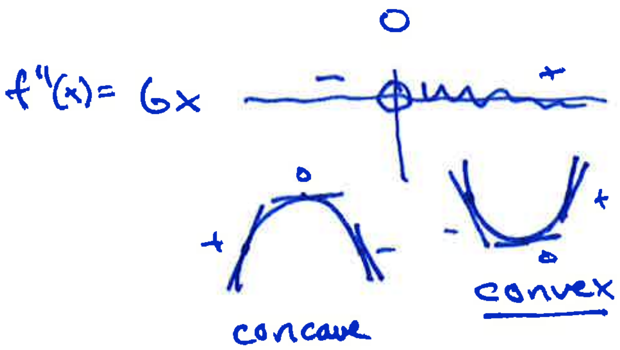
$f'(x) = 3x^2 - 2$

second order derivative of f $\longrightarrow$ $f''(x) = 6x$

Interpretation of $f''(x)$:

$$f'(x) = 3x^2 - 2$$
$$= 3(x^2 - 2/3)$$
$$= 3(x - \sqrt{2/3})(x + \sqrt{2/3})$$





$f' > 0$: graph of f increasing

$f' < 0$: graph of f decreas.

I: interval

$f''(x) \geq 0$ for all x in I: f convex in I

$f''(x) \leq 0$ — " — : f concave in I

f convex: D_f is an interval + f is convex in D_f

f concave: — " — + f is concave in D_f

2 Optimization : max/min - problems

$$\boxed{\max/\min \ f(x)}$$

Method: 1 Find stationary pts for f

= all points s.t. $f'(x) = 0$

$$\boxed{x = x^* \text{ is max/min} \implies x = x^* \text{ is a stationary pt.}}$$

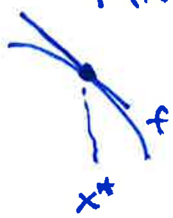
$f'(x^*) > 0$

not max/min

$f'(x^*) = 0$

max

$f'(x^*) < 0$



not max/min .

min

Conclusion:

stationary pts are candidates for max/min

not max
not min
saddle point

(2) Methods for classifying stationary points

a) Make a sign diagram for $f'(x)$.

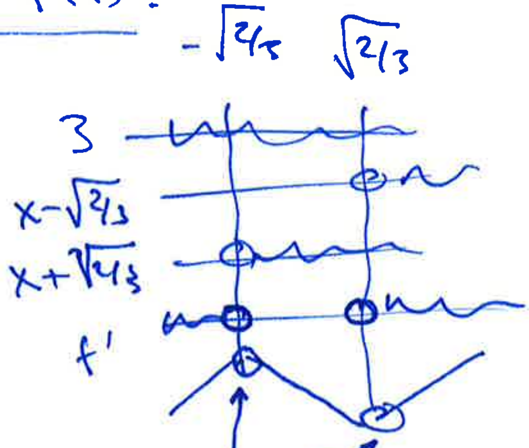
Ex: $f(x) = x^3 - 2x + 1$

$f'(x) = 3x^2 - 2 = 0$

$\frac{3x^2}{3} = \frac{2}{3}$

$x^2 = 2/3$

Stationary points $\longrightarrow$ $x = \pm\sqrt{2/3}$



$x = -\sqrt{2/3}$ local max

$x = \sqrt{2/3}$ local min

b) Second derivative test:

If $x = x^*$ stationary pt:

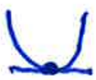
$f''(x^*) < 0$: $x = x^*$ local max

$f''(x^*) > 0$: $x = x^*$ local min

$f'' < 0$:

concave

$f'' > 0$



convex

Ex: $f''(x) = 6x$

$x = \sqrt{2/3}$: $f''(\sqrt{2/3}) = 6 \cdot \sqrt{2/3} > 0$

local min

$x = -\sqrt{2/3}$: $f''(-\sqrt{2/3}) = 6 \cdot (-\sqrt{2/3})$ < 0

local max

Ex:

$V = 500$: $\boxed{\pi r^2 \cdot h = 500}$

min A $\quad A = 2\pi r \cdot h + 2\pi r^2$

$$h = \frac{500}{\pi r^2}$$

$$A = 2\cancel{\pi r} \cdot \left(\frac{500}{\cancel{\pi} r^{\cancel{2}}}\right) + 2\pi r^2$$

$\boxed{\text{min } A(r)}$

$$A(r) = \frac{1000}{r} + 2\pi r^2 \, , \quad \underline{r > 0}$$

$$A'(r) = -\frac{1000}{r^2} + 2\pi \cdot 2r$$

$$= \underline{-\frac{1000}{r^2} + 4\pi r = 0}$$

$$4\pi r = \frac{1000}{r^2} \quad | \cdot r^2$$

$$\frac{4\pi r^3}{4\pi} = \frac{1000}{4\pi}$$

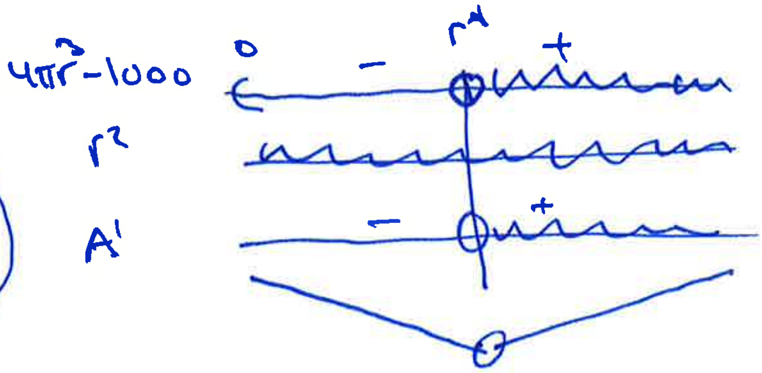
$$r^3 = \frac{250}{\pi}$$

$$r = \sqrt[3]{\frac{250}{\pi}} \simeq 4.3$$

$$r^* = \underline{\sqrt[3]{\frac{250}{\pi}}}$$

local min
global min
= min

$$= \frac{-1000 + 4\pi r \cdot r^2}{r^2} = \frac{4\pi r^3 - 1000}{r^2}$$



$$\underline{r^* = \sqrt[3]{\frac{250}{\pi}}} \qquad h^* = \frac{500}{\pi (r^*)^2} = \underline{2r^*}$$

Summary: max/min $f(x)$

① Solve $f'(x) = 0$ (first order condition) ⟶ stationary points = candidates for max/min

② For each stationary point $x = x^*$, ⟶ classify as $\begin{cases} \text{local max} \\ \text{local min} \\ \text{saddle pt.} \end{cases}$

using either

i) sign diagram for $f'(x)$

ii) second derivative test ⟶ $f''(x^*)$

Main result:

If $x = x^*$ is a (local) max or min of f, then we have either

i) $x = x^*$ is a stationary pt: $f'(x^*) = 0$

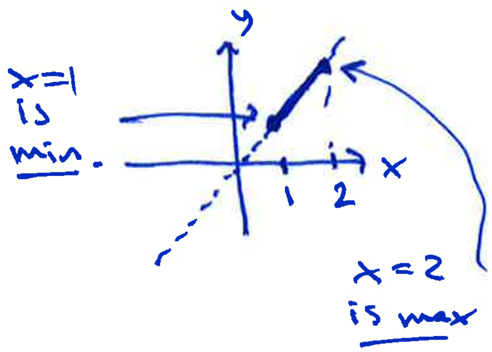
ii) $f'(x^*)$ does not exist

iii) x^* is a boundary pt.

ii), iii): exceptions

Ex: $f(x) = x$, $1 \le x \le 2$

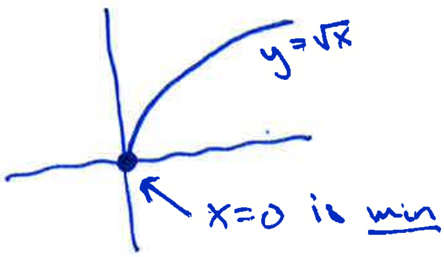
$f' = 1$ $x=1, x=2$ are boundary pts.



Ex: $f(x) = \sqrt{x}$, $x \geq 0$

$f'(x) = \frac{1}{2\sqrt{x}}$ $x=0$ is boundary point

defined for $x > 0$
not defined for $x = 0$



3 Integration

Antiderivative:

Ex: $f(x) = \underline{2x}$

$F(x) = x^2$ antiderivative of $f(x) = 2x$ $(x^2)' = 2x$

$F(x) = x^2 + 1$ $(x^2+1)' = 2x$

$F(x) = \underline{x^2 + C}$ general antiderivative

Integral = indefinite integral:

$$\int 2x \, dx = x^2 + C$$

$\int$: integration sign

dx: x is integration variable

C: constant of integration

Integration rules:

1. $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C$, $n \neq -1$
2. $\int u \pm v \, dx = \int u \, dx \pm \int v \, dx$
3. $\int c \cdot u \, dx = c \cdot \int u \, dx$
4. $\int e^x \, dx = e^x + C$
5. $\int \frac{1}{x} \, dx = \ln|x| + C$

u, v: expressions in x

c: constant

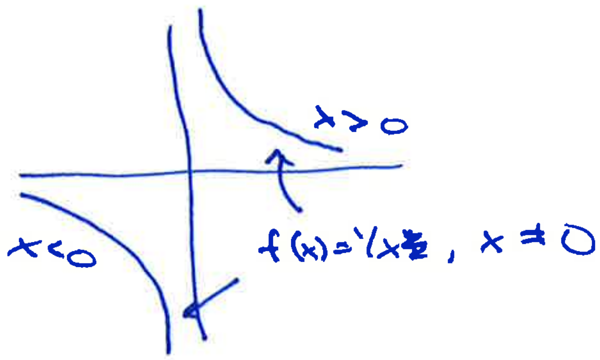
Ex: $\int 2x^3 - 4x + 7 \, dx$

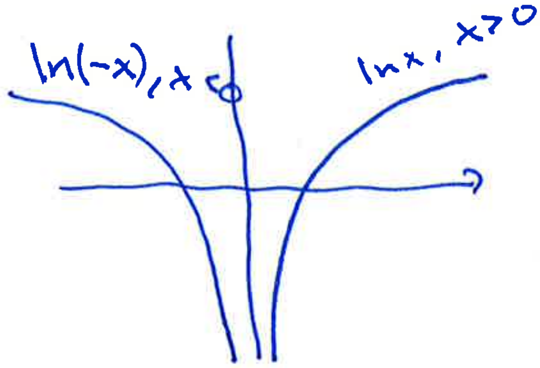
$$= 2 \cdot \int x^3 dx - 4 \int x \, dx + \int 7 \, dx$$

$$= 2 \cdot \frac{x^4}{4} - 4 \cdot \frac{x^2}{2} + 7x + C$$

$$= \underline{\frac{1}{2}x^4 - 2x^2 + 7x + C}$$

Why is $\int 1/x \, dx = \ln|x| + C$?





$$(\ln x)' = 1/x$$

$$(\ln(-x))' = \frac{1}{-x} \cdot (-1)$$

$$= 1/x$$

$$\int \frac{1}{x} dx = \begin{Bmatrix} \ln x, & x>0 \\ \ln(-x), & x<0 \end{Bmatrix} + C$$

$$= \ln|x| + C$$

Advanced integration techniques:

(a) Substitution:

Ex:

$$\int e^{1-x}\,dx = \int e^{u}\cdot\frac{du}{-1} = -\int e^{u}\,du$$

$$dx = \frac{du}{u'} \rightarrow \boxed{\begin{aligned} u &= 1-x \\ du &= u'\cdot dx \end{aligned}}$$

$$\frac{du}{dx} = u'$$

$$\frac{du}{-1} = \frac{-1\cdot dx}{-1}$$

$$\frac{du}{-1} = dx$$

$$= -e^{u} + C$$

$$= -e^{1-x} + C$$

Test: $\left(-e^{1-x}\right)' = -e^{1-x}\cdot(-1)$

$$= e^{1-x}$$

Ex:

$$\int x\cdot\ln(x^2+1)\,dx = \int \cancel{x}\cdot\ln(u)\cdot\frac{du}{2\cancel{x}}$$

$$\boxed{\begin{aligned} u &= x^2+1 \\ du &= 2x\cdot dx \end{aligned}}$$

$$= \frac{1}{2}\int \ln(u)\,du = \ldots$$

$$= \frac{1}{2}\left(u\ln u - u\right) + C$$

$$= \frac{1}{2}u\ln u - \frac{1}{2}u + C$$

$$= \frac{1}{2}(x^2+1)\ln(x^2+1) - \frac{1}{2}(x^2+1) + C$$

(b) Integration by parts (to integrate products)

$$\int u' v\,dx = u \cdot v - \int u \cdot v'\,dx$$

$$(uv)' = u'v + uv'$$

$$uv = \int u'v\,dx + \int uv'\,dx$$

$$\boxed{\int u' \cdot v\,dx = u \cdot v - \int u \cdot v'\,dx}$$

Ex: $\int x \cdot e^x\,dx = e^x \cdot x - \int e^x \cdot 1\,dx$

$$\boxed{\begin{array}{ll} u = e^x & v = x \\ u' = e^x & v' = 1 \end{array}}$$

$$= xe^x - \int e^x\,dx$$

$$= \underline{xe^x - e^x + C}$$

Ex: $\int \ln x\,dx = \int 1 \cdot \ln x\,dx = x \cdot \ln x - \int x \cdot \frac{1}{x}\,dx$

$$\boxed{\begin{array}{ll} u = x & v = \ln x \\ u' = 1 & v' = 1/x \end{array}}$$

$$= x\ln x - \int 1\,dx = \underline{x\ln x - x + C}$$

$$\boxed{\int \ln x\,dx = x\ln x - x + C}$$

Problems for Lecture 5

1. Compute the integrals

a) $\int x^2 \ln x \, dx$

b) $\int \frac{x^3 + 2x^2 + 1}{x} dx$

c) $\int x e^{-x^2} dx$

d) $\int \frac{x}{x^2+1} dx$

e) $\int \frac{1}{(2x-3)^2} dx$

f) $\int x^3 e^{-x^2} dx$

g) $\int e^{\sqrt{x}} dx$

2. Simplify the expressions using polynomial division:

a) $\frac{x^3}{x^2-1}$

b) $\frac{x^2+2x-3}{x+1}$

c) $\frac{x^3+x^2+x+1}{x+1}$

d) $\frac{x^4+1}{x-1}$

3. Compute the integrals:

a) $\int \frac{3}{2x-4} dx$

b) $\int \frac{1}{x^2+x} dx$

c) $\int \frac{x}{x^2-4x-5} dx$

d) $\int \frac{x^2}{x^2-1} dx$

e) $\int \frac{1}{x^2-4x+4} dx$

4. Find the first order partial derivatives and the Hessian:

a) $f(x,y) = x^3 - 3xy + y^3$

b) $f(x,y) = e^{xy}$

c) $f(x,y) = e^{x+2y}$

d) $f(x,y) = \sqrt{x^2+y^2+1}$

e) $f(x,y) = \ln(x^2+y^2+4)$

f) $f(x,y) = \ln(xy) - 1$

g) $f(x,y) = x\ln y - y\ln x$

Solutions for Lecture 5.

BI

1. a) $\int \underset{v'}{x^2} \underset{u}{\ln x}\, dx = \underset{v}{\tfrac{1}{3}x^3} \cdot \underset{u}{\ln x} - \int \underset{v}{\tfrac{1}{3}x^3} \cdot \underset{u'}{\tfrac{1}{x}}\, dx$

$$= \tfrac{1}{3}x^3 \ln x - \tfrac{1}{3}\int x^2 dx = \underline{\tfrac{1}{3}x^3 \ln x - \tfrac{1}{9}x^3 + C}$$

b) $\int \frac{x^3+2x^2+1}{x} dx = \int x^2 + 2x + \frac{1}{x}\, dx = \underline{\tfrac{1}{3}x^3 + x^2 + \ln|x| + C}$

c) $\int x e^{-x^2} dx = \int \cancel{x} e^{u} \cdot \frac{du}{-2\cancel{x}} = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + C$

$u = -x^2$, $du = -2x\,dx$

$$= \underline{-\tfrac{1}{2} e^{-x^2} + C}$$

d) $\int \frac{x}{x^2+1} dx = \int \frac{x}{u} \cdot \frac{du}{2x} = \frac{1}{2}\int \frac{1}{u} du = \frac{1}{2}\ln|u| + C$

$u = x^2+1$, $du = 2x\,dx$

$$= \tfrac{1}{2}\ln|x^2+1| + C$$

$$= \underline{\tfrac{1}{2}\ln(x^2+1) + C}$$

e) $\int \frac{1}{(2x-3)^2} dx = \int \frac{1}{u^2} \frac{du}{2} = \frac{1}{2}\int u^{-2} du$

$u = 2x-3$, $du = 2\,dx$

$$= \tfrac{1}{2}\left(-\tfrac{1}{u}\right) + C$$

$$= \underline{-\tfrac{1}{2} \cdot \tfrac{1}{2x-3} + C}$$

f) $\int x^3 e^{-x^2} dx = \int x^3 e^u \frac{du}{-2x} = \int -\frac{1}{2}x^2 e^u du$

$u = -x^2$, $du = -2x\,dx$

$$= -\tfrac{1}{2}\int (-u) e^u du$$

$x^2 = -u$

$$= \tfrac{1}{2}\int u e^u du \underset{\text{int. by parts}}{=} u e^u - \int 1 \cdot e^u du = u e^u - e^u + C$$

$$= \underline{-x^2 e^{-x^2} - e^{-x^2} + C}$$

g) $\int e^{\sqrt{x}}\,dx = \int e^u \cdot 2\sqrt{x}\,du$

$u=\sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$= \int e^u \cdot 2u\,du$

$\sqrt{x}=u$

$= \int 2ue^u\,du = 2ue^u - \int 2e^u\,du$

int. by parts

$= 2ue^u - 2e^u + C = \underline{2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C}$

2. a) $x^3 : x^2-1 = x \Rightarrow \underline{\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}}$

$-(x^3-x)$

x

b) $x^2+2x-3 : x+1 = x+1 \Rightarrow \frac{x^2+2x-3}{x+1} = \underline{x+1+\frac{-4}{x+1}}$

$-(x^2+x)$

$x-3$

$-(x+1)$

-4

c) $x^3+x^2+x+1 : x+1 = x^2+1 \Rightarrow \frac{x^3+x^2+x+1}{x+1} = \underline{x^2+1}$

$-(x^3+x^2)$

$x+1$

$x+1$

0

d) $x^4+1 : x-1 = x^3+x^2+x+1 \Rightarrow \frac{x^4+1}{x-1} = \underline{x^3+x^2+x+1+\frac{2}{x-1}}$

$-(x^4-x^3)$

x^3+1

$-(x^3-x^2)$

x^2+1

$-(x^2-x)$

$x+1$

$-(x-1)$

2

3. a) $\int \frac{3}{2x-4}\,dx = \int \frac{3}{u}\frac{du}{2} = \frac{3}{2}\int \frac{1}{u}\,du$

$u = 2x-4$
$du = 2dx$

$= \frac{3}{2}\ln|2x-4| + C$

b) $\int \frac{1}{x^2+x}\,dx \underset{\uparrow}{=} \int \frac{1}{x} + \frac{-1}{x+1}\,dx = \ln|x| - \ln|x+1| + C$

$\frac{1}{x^2+x} = \frac{1}{x\cdot(x+1)} = \frac{A}{x} + \frac{B}{x+1} \quad \Big| \cdot (x+1)x$

$1 = A\cdot(x+1) + Bx$

$= \underset{\overset{\shortparallel}{0}}{(A+B)}x + \underset{\overset{\shortparallel}{1}}{(A)}$

$A = 1$, $B = -1$

$= \ln\left|\frac{x}{x+1}\right| + C$

rules for logarithms

c) $\int \frac{x}{x^2-4x-5}\,dx \underset{\uparrow}{=} \int \frac{5/6}{x-5} + \frac{1/6}{x+1}\,dx$

$x^2-4x-5 = (x-5)(x+1)$

$\frac{x}{x^2-4x-5} = \frac{A}{x-5} + \frac{B}{x+1} \quad \Big| \cdot (x-5)(x+1)$

$x = A(x+1) + B(x-5)$

$= \underset{\overset{\shortparallel}{1}}{(A+B)}x + \underset{\overset{\shortparallel}{0}}{(A-5B)}$

$A+B = 1$
$A-5B = 0 \Rightarrow A = 5B$

$5B + B = 1$

$6B = 1$

$A = 5/6$ $\quad B = 1/6$

$= \frac{5}{6}\ln|x-5| + \frac{1}{6}\ln|x+1| + C$

$= \frac{1}{6}\left(\ln|x-5|^5 + \ln|x+1|\right) + C$

$= \frac{1}{6}\ln\left|(x-5)^5(x+1)\right| + C$

d) $\int \frac{x^2}{x^2-1}\,dx = \int 1 + \frac{1}{x^2-1}\,dx = x + \int \frac{1}{(x-1)(x+1)}\,dx$

$$\begin{array}{l} x^2 : x^2-1 = 1 \\ -(x^2-1) \\ \hline 1 \end{array}$$

$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} \quad | \cdot (x-1)(x+1)$

$1 = A(x+1) + B(x-1)$

$= \underbrace{(A+B)}_{0}x + \underbrace{(A-B)}_{1}$

$A + B = 0$

$A - B = 1$

$2A = 1 \longrightarrow \underline{A = 1/2},\ \underline{B = -1/2}$

$= x + \int \frac{1/2}{x-1} + \frac{-1/2}{x+1}\,dx = x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$

$= x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$

e) $\int \frac{1}{x^2-4x+4}\,dx = \int \frac{1}{(x-2)^2}\,dx = \int \frac{1}{u^2}\,du = -\frac{1}{u} + C$

$u = x-2$

$du = 1 \cdot dx$

$= -\frac{1}{x-2} + C$

4.

a) $f = x^3 - 3xy + y^3$

$f'_x = 3x^2 - 3y$

$f'_y = -3x + 3y^2$

$f''_{xx} = 6x$ $\quad f''_{xy} = -3$

$f''_{yy} = 6y$

know that $f''_{yx} = f''_{xy} = -3$

$H(f) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$

b) $f = e^{xy}$

$f'_x = e^{xy} \cdot (xy)'_x = ye^{xy}$

$f'_y = e^{xy} \cdot (xy)'_y = xe^{xy}$

$f''_{xx} = y \cdot ye^{xy} = y^2 e^{xy}$

$f''_{xy} = 1 \cdot e^{xy} + y \cdot e^{xy} \cdot x = (xy+1)e^{xy}$

$f''_{yy} = x \cdot x e^{xy} = x^2 e^{xy}$

$H(f) = \begin{pmatrix} y^2 e^{xy} & (xy+1)e^{xy} \\ (xy+1)e^{xy} & x^2 e^{xy} \end{pmatrix}$

$= \begin{pmatrix} y^2 & xy+1 \\ xy+1 & x^2 \end{pmatrix} \cdot e^{xy}$

c) $f = e^{x+2y}$

$f'_x = e^{x+2y}$

$f'_y = 2e^{x+2y}$

$f''_{xx} = e^{x+2y}$ $\quad f''_{xy} = 2e^{x+2y}$

$f''_{yy} = 4e^{x+2y}$

$H(f) = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} e^{x+2y}$

d) $f = \sqrt{x^2+y^2+1}$

$f'_x = \frac{1}{2\sqrt{x^2+y^2+1}} \cdot 2x = \frac{2x}{2\sqrt{--}} = \frac{x}{\sqrt{x^2+y^2+1}}$

$f'_y = \frac{1}{2\sqrt{\sim}} \cdot 2y = \frac{2y}{2\sqrt{--}} = \frac{y}{\sqrt{x^2+y^2+1}}$

$u = x^2+y^2+1$: $\quad f'_x = \frac{x}{\sqrt{u}} \quad f'_y = \frac{y}{\sqrt{u}}$

$$f''_{xx} = \frac{1\cdot\sqrt{u} - x\cdot\frac{1}{2\sqrt{u}}\cdot\cancel{2}x}{u} = \frac{\sqrt{u} - \frac{x^2}{\sqrt{u}}}{u}\cdot\frac{\sqrt{u}}{\sqrt{u}} = \frac{u-x^2}{u\sqrt{u}} = \frac{\cancel{x^2}+y^2+1-\cancel{x^2}}{(x^2+y^2+1)^{3/2}}$$

$$= \frac{y^2+1}{(x^2+y^2+1)^{3/2}}$$

$$f''_{xy} = \frac{0\cdot\sqrt{u} - x\cdot\frac{1}{\cancel{2}\sqrt{u}}\cdot\cancel{2}y}{u} = \frac{-xy}{u\sqrt{u}} = \frac{-xy}{(x^2+y^2+1)^{3/2}}$$

$$f''_{yy} = \frac{x^2+1}{(x^2+y^2+1)^{3/2}}$$

By symmetry: $f(x,y) = f(y,x)$

so $f''_{yy}(x,y) = f''_{xx}(y,x)$

(i.e. we swap x and y)

$$H(f) = \frac{1}{(x^2+y^2+1)^{3/2}}\cdot\begin{pmatrix} y^2+1 & -xy \\ -xy & x^2+1 \end{pmatrix}$$

e) $f = \ln(x^2+y^2+4) = \ln(u)$, $u = x^2+y^2+4$

$$f'_x = \frac{2x}{x^2+y^2+4} \qquad f''_{xx} = \frac{2(x^2+y^2+4)-2x(2x)}{u^2} = \frac{-2x^2+2y^2+4}{(x^2+y^2+4)^2}$$

$$f'_y = \frac{2y}{x^2+y^2+4} \qquad f''_{xy} = \frac{0\cdot\cancel{x} - 2x\cdot 2y}{u^2} = \frac{-4xy}{(x^2+y^2+4)^2}$$

$$f''_{yy} = \frac{2x^2-2y^2+4}{(x^2+y^2+4)^2}$$

by symmetry

$$H(f) = \frac{1}{(x^2+y^2+4)}\cdot\begin{pmatrix} -2x^2+2y^2+4 & -4xy \\ -4xy & 2x^2-2y^2+4 \end{pmatrix}$$

$f = \ln(xy) - 1 = \ln x + \ln y - 1$

f) $f'_x = \frac{1}{x}$ $\qquad f''_{xx} = -1/x^2 \qquad f''_{xy} = 0$

$f'_y = \frac{1}{y} \qquad\qquad f''_{yy} = -1/y^2$

$$H(f) = \begin{pmatrix} -\frac{1}{x^2} & 0 \\ 0 & -\frac{1}{y^2} \end{pmatrix}$$

g) $f = x\ln y - y\ln x$

$f'_x = \ln y - \frac{y}{x}$

$f'_y = \frac{x}{y} - \ln x$

$f''_{xx} = -y \cdot (-1/x^2) = \frac{y}{x^2}$

$f''_{xy} = \frac{1}{y} - \frac{1}{x}$

$f''_{yy} = x \cdot (-1/y^2) = -\frac{x}{y^2}$

$$H(f) = \begin{pmatrix} y/x^2 & 1/y - 1/x \\ 1/y - 1/x & -x/y^2 \end{pmatrix}$$