FORK1005
Solutions for Exercises 6

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2 Direct Substitution

Solution 2.1. We have the budget constraint

$$4x + 3y = 600,$$

and the maximizing function

$$U(x, y) = xy.$$

We rearrange the budget constraint to put y in terms of x:

$$y = \frac{600 - 4x}{3}$$

Then we can substitute this value of y into $U(x, y)$ to get a function of only one variable, x:

$$f(x) := U\left(x, \frac{600 - 4x}{3}\right) = x\frac{600 - 4x}{3} = 200x - \frac{4}{3}x^2.$$

We want to maximize f so we differentiate and set equal to zero:

$$f'(x) = 200 - \frac{8}{3}x = 0 \qquad \Longrightarrow \qquad 200 = \frac{8}{3}x$$

so

$$x = 75 \qquad \Longrightarrow \qquad y = \frac{600 - 4x}{3} = \frac{600 - 4 \cdot 75}{3} = 100.$$

So we have the solution

$$x = 75, \qquad y = 100.$$

To check that it is a maximizer, we take the second derivative:

$$f''(x) = -\frac{8}{3} < 0$$

so by the second derivative test, $x = 75$ maximizes $f(x)$.

3 The Lagrange Multiplier Method

Solution 3.1. We have the budget constraint

$$4x + 3y = 600,$$

and the maximizing function

$$U(x, y) = xy.$$

This gives us the Lagrangian function

$$\mathcal{L}(x, y, \lambda) = xy + \lambda(4x + 3y - 600).$$

Taking partial derivatives and setting to zero, we get the system of equations

$$\begin{aligned} \mathcal{L}'_x(x, y, \lambda) &= y + 4\lambda = 0, \\ \mathcal{L}'_y(x, y, \lambda) &= x + 3\lambda = 0, \\ \mathcal{L}'_\lambda(x, y, \lambda) &= 4x + 3y - 600 = 0 \end{aligned}$$

Rearranging the first two equations, we get

$$-\lambda = \frac{y}{4} \qquad \text{and} \qquad -\lambda = \frac{x}{3}$$

and combining these, we get

$$\frac{y}{4} = \frac{x}{3} \qquad \Longrightarrow \qquad 4x = 3y.$$

We plug in $4x$ for $3y$ into the budget constraint to get

$$4x + 4x = 600 \qquad \Longrightarrow \qquad x = 75.$$

This means that

$$y = \frac{4}{3}x = \frac{4}{3}75 = 100.$$

So the maximizer (or minimizer) of the constraint problem is

$$x = 75, \quad y = 100.$$

We can tell that it's a maximizer by exploring other values of x and y that satisfy the constraint. For example $(x, y) = (0, 200)$ satisfies the constraint but $U(0, 200) = 0$. This tells us that $(75, 100)$ must be a maximizer and not a minimizer.

Solution 3.2. We have the budget constraint

$$3x + 2y = 300,$$

and the maximizing function

$$U(x, y) = x^{1/2}y^{1/2}.$$

This gives us the Lagrangian function

$$\mathcal{L}(x,y,\lambda) = x^{1/2}y^{1/2} + \lambda(3x + 2y - 300).$$

Taking partial derivatives and setting to zero, we get the system of equations

$$\begin{aligned}
\mathcal{L}'_x(x,y,\lambda) &= \frac{y^{1/2}}{2x^{1/2}} + 3\lambda = 0,\\
\mathcal{L}'_y(x,y,\lambda) &= \frac{x^{1/2}}{2y^{1/2}} + 2\lambda = 0,\\
\mathcal{L}'_\lambda(x,y,\lambda) &= 3x + 2y - 300 = 0
\end{aligned}$$

Rearranging the first two equations, we get

$$-\lambda = \frac{y^{1/2}}{6x^{1/2}} \qquad \text{and} \qquad -\lambda = \frac{x^{1/2}}{4y^{1/2}}$$

and combining these, we get

$$\begin{aligned}
\frac{y^{1/2}}{6x^{1/2}} &= \frac{x^{1/2}}{4y^{1/2}}\\
6x &= 4y\\
3x &= 2y.
\end{aligned}$$

We plug in $3x$ for $2y$ into the budget constraint to get

$$2y + 2y = 300, \qquad \Longrightarrow \qquad y = 75.$$

This means that

$$3x = 2y = 2 \cdot 75 = 150 \qquad \Longrightarrow \qquad x = 50.$$

So the maximizer (or minimizer) of the constraint problem is

$$x = 50, \quad y = 75.$$

It must be a maximizer, since plugging in any other values that satisfy the constraint will give you a smaller value for $U(x,y)$.

Solution 3.3. We have the constraint

$$x^2 + 2y^2 = 400,$$

and the maximizing function

$$f(x,y) = x^2 + y^2 - 4x + 30.$$

This gives us the Lagrangian function

$$\mathcal{L}(x,y,\lambda) = x^2 + y^2 - 4x + 30 + \lambda(x^2 + 2y^2 - 400)$$

Taking partial derivatives and setting to zero, we get the system of equations

$$\begin{aligned}
\mathcal{L}'_x(x,y,\lambda) &= 2x - 4 + 2\lambda x = 0,\\
\mathcal{L}'_y(x,y,\lambda) &= 2y + 4\lambda y = 0,\\
\mathcal{L}'_\lambda(x,y,\lambda) &= x^2 + 2y^2 - 400 = 0
\end{aligned}$$

We cannot rearrange the first two equations to get λ in terms of x and y without dividing by x and y, which means that we must assume that x and y are not equal to zero. This gives us three scenarios:

1. $x = 0$ and $y \neq 0$
2. $x \neq 0$ and $y = 0$
3. $x \neq 0$ and $y \neq 0$

First scenario: If $x = 0$, y must satisfy the budget constraint

$$\begin{aligned}
2y^2 &= 400\\
y &= \pm\sqrt{200}.
\end{aligned}$$

So we have the solutions

$$(x,y) = (0,\sqrt{200}) \quad \text{and} \quad (x,y) = (0,-\sqrt{200}).$$

Plugging either of these into $f(x,y)$, we get

$$f(0,\pm\sqrt{200}) = 200 + 30 = 230.$$

Second scenario: If $y = 0$, x must satisfy the budget constraint

$$\begin{aligned}
x^2 &= 400\\
x &= \pm 20.
\end{aligned}$$

So we have the solutions

$$(x,y) = (20,0) \quad \text{and} \quad (x,y) = (-20,0).$$

Plugging $(20,0)$ into f we get

$$400 - 80 + 30 = 350.$$

Plugging $(-20,0)$ into f we get

$$400 + 80 + 30 = 510.$$

Third scenario: If $x, y \neq 0$, we can rearrange the system of equations to get

$$-\lambda = \frac{2x-4}{2x} \qquad \text{and} \qquad -\lambda = \frac{2y}{4y} = \frac{1}{2}.$$

Combining these equations, we get

$$\begin{aligned} \frac{2x-4}{2x} &= \frac{1}{2} \\ 2x-4 &= x \\ x &= 4. \end{aligned}$$

y must then satisfy the budget constraint

$$16 + 2y^2 = 400 \qquad \Longrightarrow \qquad y = \pm\sqrt{192}.$$

So this gives us the solutions

$$(x,y) = (4, \sqrt{192}) \quad \text{and} \quad (x,y) = (4, -\sqrt{192}).$$

Plugging these into f, we get

$$f(4, \pm\sqrt{192}) = 4^2 + \sqrt{192}^2 - 4 \cdot 4 + 30 = 222.$$

Out of all the possible solutions we found, $(x,y) = (-20,0)$ gave us the largest value for f, so the solution is

$$(x,y) = (-20,0).$$

Solution 3.4. We have the budget constraint

$$4x + 3y = 50$$

and the minimizing function

$$f(x,y) = x^2 + y^2$$

This gives us the Lagrangian function

$$\mathcal{L}(x,y,\lambda) = x^2 + y^2 + \lambda(4x + 3y - 50).$$

Taking partial derivatives and setting to zero, we get the system of equations

$$\begin{aligned} \mathcal{L}'_x(x,y,\lambda) &= 2x + 4\lambda = 0, \\ \mathcal{L}'_y(x,y,\lambda) &= 2y + 3\lambda = 0, \\ \mathcal{L}'_\lambda(x,y,\lambda) &= 4x + 3y - 50 = 0 \end{aligned}$$

Rearranging the first two equations, we get

$$-\lambda = \frac{x}{2} \qquad \text{and} \qquad -\lambda = \frac{2y}{3}$$

and combining these, we get

$$x = \frac{4}{3}y.$$

We plug in $4y/3$ for x into the budget constraint to get

$$\begin{aligned} 4\frac{4y}{3} + 3y &= 50 \\ \frac{16y + 9y}{3} &= 50 \\ y &= 6. \end{aligned}$$

This means that

$$x = \frac{4}{3} \cdot 6 = 8.$$

So the minimizer (or maximizer) of the constraint problem is

$$x = 8, \quad y = 6.$$

Plugging in any other value that satisfies the constraint will give you a larger value of f, so we know that this is a minimizer.