FORK1005
Solutions for Exercises 3

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1 Introduction to Differentiation

Solution 1.1. $f'(-2) = -4$ and $f'(1) = 2$.

Solution 1.2. From left to right: $+$, $-$, 0, $+$.

2 Differentiation

Solution 2.1.

(a) $6x$

(b) $4x^3 - 3x^2 + \dfrac{1}{2}$

(c)
$$\frac{(x+h)^3 - x^3}{h} = \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = 3x^2 + 3xh + h^2$$
so
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 3x^2 + 3xh + h^2 = 3x^2.$$

(d) $2ax + b$

(e) $\dfrac{1}{2\sqrt{x}}$

(f) $-\dfrac{1}{x^2}$

(g) $\dfrac{1}{3x^{2/3}}$

Solution 2.2.

(a) $D'(P) = -\beta$

(b) $C'(x) = 2qx$

Solution 2.3.

(a) 3

(b) $f'(x) = 2x$ so $f'(1) = 2$

(c) $f'(x) = -\dfrac{3}{x^2}$ so $f'(3) = -\frac{1}{3}$

(d) $f'(x) = 4x^3$ so $f'(1) = 4$.

Solution 2.4.

(a) $f'(x)$

(b) $f'(x)$

(c) $4f'(x)$

(d) $-\dfrac{f'(x)}{5}$

(e) $\dfrac{Af'(x)}{C}$

Solution 2.5.

(a) $8\pi r$

(b) $(b+1)Ay^b$

(c) $-\dfrac{5}{2A^{7/2}}$

Solution 2.6.

$$f'(a) = \lim_{x\to a} \frac{x^2 - a^2}{x - a} = \lim_{x\to a} \frac{(x-a)(x+a)}{x-a} = \lim_{x\to a} x + a = 2a.$$

2.1 Table of Derivatives and Rules

Solution 2.7.

(a) $(3x^2 - 1)(5x^4 + x^2) + (x^3 - x)(20x^3 + 2x) = 35x^6 - 20x^4 - 3x^2$

(b) $3\left(\dfrac{1}{x^2} + \dfrac{1}{x}\right) + (3x+1)\left(-\dfrac{2}{x^3} - \dfrac{1}{x^2}\right) = -\dfrac{2}{x^3} - \dfrac{4}{x^2}$

(c) $nt^{n-1}(a\sqrt{t} + b) + \dfrac{at^n}{2\sqrt{t}} = at^{n-1/2}(n - 1/2) + nbt^{n-1}$

(d) $-\dfrac{1}{(x-2)^2}$

(e) $\dfrac{3}{2x^{3/2} + 4x + 2\sqrt{x}}$

Solution 2.8.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{C(x)}{x}\right) = \frac{C'(x)x - C(x)}{x^2}.$$

2.2 Chain Rule

Solution 2.9.

(a) $-15x^2(1-x^3)^4$

(b) $\dfrac{-70(2x+4)}{(x^2+4x+5)^8}$

(c) $50(3x^2+2x)(x^3+x^2)^{49}$

(d) $\dfrac{4}{3(x+3)^{4/3}(x-1)^{2/3}}$

(e) $\dfrac{x}{\sqrt{x^2+1}}$

3 Applications of the Derivative

3.1 Increasing vs Decreasing Functions

Solution 3.1.

(a) $f'(x) = 3$ so $f'(1) = 3$ and f is increasing.

(b) $f'(x) = 2x$ so $f'(1) = 2$ and f is increasing.

(c) $f'(x) = 2x - 4$ so $f'(1) = -2$ and f is decreasing.

(d) $f'(x) = 3x^2 - 18x + 15$ so $f'(1) = 0$ and f is stationary.

(e) $f'(x) = e^x - 2x$ so $f'(1) = e - 2 > 0$ and f is increasing.

(f) $f'(x) = 1/x - 1$ so $f'(1) = 0$ and f is stationary.

Solution 3.2.

$$\begin{aligned} f'(x) &= -x^2 + 4x - 3 \\ &= -(x-1)(x-3). \end{aligned}$$

A sign diagram analysis will tell you that

$$-(x-1)(x-3) > 0$$

if and only if $1 < x < 3$. So f is increasing whenever $1 < x < 3$, and f is decreasing when $x > 3$ or $x < 1$.

3.2 Locating Maxima and Minima

Solution 3.3.

(a) $f'(x) = 2x$ so only stationary point is $x = 0$.

(b) $f'(x) = 1/x - 1$ so only stationary point is $x = 1$.

(c) $f'(x) = 2x - 1$ so only stationary point is $x = 1/2$.

(d) $f'(x) = 3x^2 + 6x - 9 = 3(x+3)(x-1)$ so stationary points are $x = -3, 1$.

(e) $f'(x) = (2x-1)g'(x^2 - x)$. Stationary points are those where either $(2x-1) = 0$ or $g'(x^2 - x) = 0$. One stationary point is $x = 1/2$. Since 2 is the only stationary point of g, we need to solve the equality $x^2 - x = 2$.

$$\begin{aligned} x^2 - x &= 2 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ x &= -1,\, 2 \end{aligned}$$

So we have the three stationary points $x = -1, 1/2$ and 2.

3.3 Classifying Stationary Points

Solution 3.4. $f'(x) = 10x^4 - 9x^2 + 2$ and $f''(x) = 40x^3 - 18x$.

Solution 3.5.

(a) Stationary points: 0. $f''(x) = 2 > 0$ so it is a minimum.

(b) Stationary points: -4. $f''(x) = -2 < 0$ so it is a maximum.

(c) Stationary points: 7, -2. $f''(x) = 12x - 30$ so $x = -2$ is a maximum and $x = 7$ is a minimum.

(d) Stationary points: 3. $f''(x) = 3e^x - e^3$ so $f''(3) > 0$ so it is a minimum.

