FORK1005
Exercises for Lecture 4

July 30, 2015

2 Integration

2.1 Antiderivative

Exercise 2.1. For a company that produces x units of a good, their cost function is denoted by $C(x)$. The derivative of a cost function $C'(x)$ is called the *marginal cost function*. Suppose that a marginal cost function is given by

$$C'(x) = 2x^2 + 2x - 5.$$

Furthermore, suppose that the company's fixed cost is 100 (i.e., even if they produce 0 goods, they have a fixed cost of 100). Find the cost function $C(x)$.

Exercise 2.2. Calculate the antiderivative of

(a) $f(x) = 6x$

(b) $f(x) = 9x^2$

(c) $f(x) = x^4$

(d) $f(x) = \dfrac{1}{x^3}$

(e) $f(x) = \sqrt{x}$.

(f) $f(x) = e^{ax}$.

Exercise 2.3. Verify that the antiderivative of $\ln x$ is $x \ln x - x + C$.

2.2 Integral

Exercise 2.4. Compute the following integrals:

(a) $\displaystyle\int 3x + x^2 - 5x^4 \,\mathrm{d}x$

(b) $\displaystyle\int (x+3)^2 \,\mathrm{d}x$

2.3 Integration Rules

Exercise 2.5. Compute the following integrals:

(a) $\displaystyle\int 3e^x \,\mathrm{d}x$

(b) $\displaystyle\int \frac{4}{x} \,\mathrm{d}x$

(c) $\displaystyle\int x^6 + 2e^x \,\mathrm{d}x$

(d) $\displaystyle\int \frac{2}{x^4} - \frac{1}{x} \,\mathrm{d}x$

Exercise 2.6. Calculate

$$\int \left(\frac{3}{x} - 8e^{-4x} \right) \mathrm{d}x.$$

3 Integration Techniques

3.1 Integration by Parts

Exercise 3.1. By setting $u' = e^{4x}$ and $v = x$, solve the following integral using integration by parts:

$$\int xe^{4x} \,\mathrm{d}x.$$

Exercise 3.2. By setting $u' = \sqrt{x-1}$ and $v = 2x$, solve the following integral using integration by parts:

$$\int 2x\sqrt{x-1} \,\mathrm{d}x.$$

Exercise 3.3. Using integration by parts, calculate these integrals:

(a) $\int xe^{-x} \,\mathrm{d}x$

(b) $\int \ln x \,\mathrm{d}x$

3.2 Integration by Substitution

Exercise 3.4. By setting $u = x - 4$, solve the following integral by substitution:

$$\int (x-4)^6 \,\mathrm{d}x.$$

Exercise 3.5. By setting $u = x^3 + 13$, solve the following integral by substitution:

$$\int 3x^2(x^3+13)^{20} \,\mathrm{d}x.$$

Exercise 3.6. By setting $u = 3 - x$, solve the following integral by substitution:

$$\int \frac{1}{3-x} \,\mathrm{d}x.$$

Exercise 3.7. By setting $u = \sqrt{1+x^2}$, solve the following integral by substitution:

$$\int 2x\sqrt{1+x^2}\,\mathrm{d}x.$$

Exercise 3.8. Using integration by substitution, calculate these integrals:

(a) $\displaystyle\int 2x(x^2+10)^{50}\,\mathrm{d}x$

(b) $\displaystyle\int xe^{-cx^2}\,\mathrm{d}x$

(c) $\displaystyle\int (4x^3-6x^2)(x^4-2x^3+5)^7\,\mathrm{d}x$

3.3 Integration by Partial Fractions

Exercise 3.9. Using integration by partial fractions, calculate these integrals:

(a) $\displaystyle\int \frac{x}{x^2+2x-3}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{1+x}{x^2-4x-12}\,\mathrm{d}x$

FORK1005
Exercises for Lecture 5

August 10, 2015

2 Partial Differentiation

Exercise 2.1. Compute all partial derivatives of the following functions:

(a) $f(x, y) = 2x + 4y$

(b) $f(x, y) = 3x + 5xy + 2y$

(c) $f(x, y) = 5x^2y^3$

(d) $f(x, y) = 3e^xy^2 - x^2e^y$

(e) $f(x, y, z) = 4xy^2z^3 - xy$

(f) $f(x, y) = e^{xy} - y\ln(x^2)$

3 First-Order Conditions

Exercise 3.1. Locate all stationary points of the following functions. For each stationary point, try to guess whether they are a maximum, minimum or neither.

(a) $f(x, y) = 3x^3 - y^2$

(b) $f(x, y) = (x - 2)^2 + y^2$

(c) $f(x, y) = x^2 - 2xy + y^2$

(d) $f(x, y) = \sqrt{x^2 + y^2 + 1}$

(e) $f(x, y) = e^{x+y}$

(f) $f(x, y) = e^{x^2+y^2}$

(g) $f(x, y) = \ln(x^2 + 1) - xy$

(h) $f(x, y, z) = (x - 1)^3 + (y + 2)^2 + (z - 5)$

4 Second-Order Derivatives

4.2 Second-Order Partial Derivatives

Exercise 4.1. Compute the second-order partial derivatives f''_{xx}, f''_{xy} and f''_{yy} of the following functions:

(a) $f(x, y) = x^2 - 5y^2$

(b) $f(x, y) = 3x^2 - 4xy - y^2$

(c) $f(x, y) = x^7 - 5x^2y^3 + y^4$

(d) $f(x, y) = 4x^3y - xe^y$

(e) $f(x, y) = y^2 \ln(x)$

(f) $f(x, y) = \sqrt{x^2 + y^2}$

5 Second Partial Derivative Test

Exercise 5.1. For the following functions, locate all stationary points, and compute the Hessian matrix. Then, for each stationary point, apply the second partial derivative test to determine whether the point is a maximum, minimum or a saddle point:

(a) $f(x, y) = 3x^2 + 5y^2$

(b) $f(x, y) = -x^2 + 4y^2$

(c) $f(x, y) = x^3 - 4y^2$

(d) $f(x, y) = x^3 - 4xy - 2y^2$

8 Convex vs Concave Functions

Exercise 8.1. Compute the Hessian matrix of the following functions, and thus determine whether each function is convex, concave or neither:

(a) $f(x, y) = (x + 2y)^4$

(b) $f(x, y) = e^{-(x+y)^2}$

(c) $f(x, y) = x(x - 2)(x + 3) - (y - 2)(y + 5)$

9 Extra Practice Problems

Exercise 9.1. A company produces and sells two types of goods, A and B. When they produce x tons of A, they sell it at the price $p(x) = 100 - 4x$ per ton, and when they produce y tons of B, they sell it at the price $q(y) = 80 - 2y$ per ton. Their cost function is given by $C(x, y) = x^2 + 3y^2 + 2xy$.

(a) Assuming that they sell everything they produce, deduce the company's profit function $P(x, y)$ from p, q and C.

(b) They want to maximize their profits. Find the unique stationary point of their profit function, and check that it is a local maximum. Is it also a global maximum?

