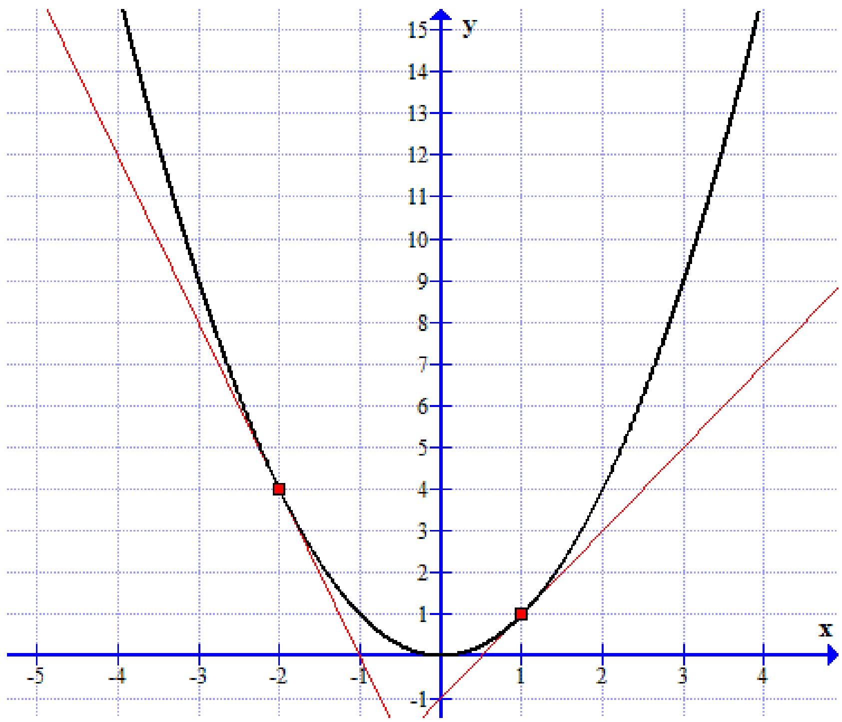
FORK1005
Exercises for Lecture 3

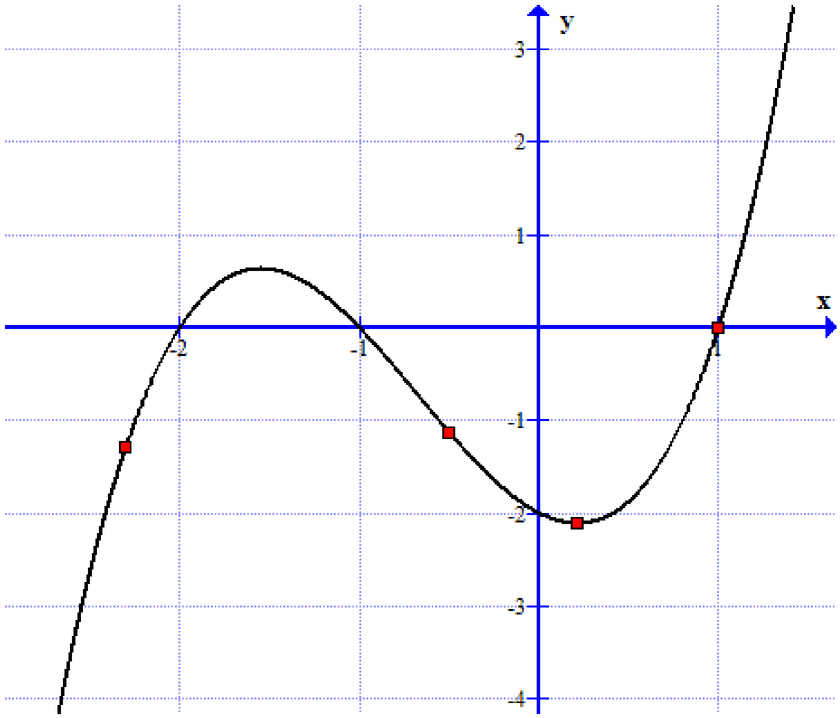
July 29, 2015

1 Introduction to Differentiation

Exercise 1.1. Below is the graph of $f(x) = x^2$. Using the red tangent lines, evaluate $f'(-2)$ and $f'(1)$. In other words, determine the slope at each of the points $(-2, 4)$ and $(1, 1)$.



Exercise 1.2. For each of the four coordinates marked in red, determine the sign (plus or minus) of the derivative $f'(x)$ at that point.



2 Differentiation

Exercise 2.1. Derivate the following functions:

(a) $f(x) = 3x^2$

(b) $f(x) = x^4 - x^3 + \frac{1}{2}x + 95$

(c) $f(x) = x^3$ from definition.

(d) $f(x) = ax^2 + bx + c$

(e) $f(x) = \sqrt{x}$

(f) $f(x) = \dfrac{1}{x}$

(g) $f(x) = x^{1/3}$

Exercise 2.2.

(a) The demand function for a product with price P is given by the formula $D(P) = \alpha - \beta P$. Find $\dfrac{\mathrm{d}D}{\mathrm{d}P}$.

(b) The cost of producing x units of the product is given by the formula $C(x) = p + qx^2$. Find $C'(x)$, the marginal cost.

Exercise 2.3. Find the slope of the tangent to the graph of f at the given coordinates:

(a) $f(x) = 3x - 2$ at $(0, -2)$

(b) $f(x) = x^2 + 1$ at $(1, 2)$

(c) $f(x) = \dfrac{3}{x} + 4$ at $(3, 5)$

(d) $f(x) = x^4 - 2$ at $(1, -1)$

Exercise 2.4. Write the derivatives of the following functions in terms of $f'(x)$. (For example, $(2f(x))' = 2f'(x)$)

(a) $5 + f(x)$

(b) $f(x) - \frac{1}{2}$

(c) $4f(x)$

(d) $-\dfrac{f(x)}{5}$

(e) $\dfrac{Af(x) + B}{C}$

Exercise 2.5. Compute the following:

(a) $\dfrac{\mathrm{d}}{\mathrm{d}r}\left(4\pi r^2\right)$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}y}\left(Ay^{b+1}\right)$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}A}\left(\dfrac{1}{A^2\sqrt{A}}\right)$

Exercise 2.6. An equivalent formulation of the definition of the derivative is the following:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Use this to find $f'(a)$ when $f(x) = x^2$.

2.3 Table of Derivatives and Rules

Exercise 2.7. Calculate derivatives for the following functions:

(a) $h(x) = (x^3 - x) \cdot (5x^4 + x^2)$

(b) $f(x) = (3x + 1)\left(\dfrac{1}{x^2} + \dfrac{1}{x}\right)$

(c) $x(t) = t^n(a\sqrt{t} + b)$

(d) $F(x) = \dfrac{3x - 5}{x - 2}$

(e) $f(x) = \dfrac{\sqrt{x} - 2}{\sqrt{x} + 1}$

Exercise 2.8. Let $C(x)$ be the total cost of producing x units of a commodity. Then the average cost of producing x units is given by $\dfrac{C(x)}{x}$. Write $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{C(x)}{x}\right)$ in terms of C' and C.

2.4 Chain rule

Exercise 2.9.

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when $y = u^5$ and $u = 1 - x^3$.

(b) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \dfrac{10}{(x^2+4x+5)^7}$.

(c) Differentiate $y = (x^3 + x^2)^{50}$.

(d) Differentiate $y = \left(\dfrac{x-1}{x+3}\right)^{1/3}$.

(e) Differentiate $y = \sqrt{x^2+1}$.

3 Applications of the Derivative

3.1 Increasing vs Decreasing Functions

Exercise 3.1. Compute the derivatives of the following functions, and thus determine whether they are increasing, decreasing or are stationary at $x = 1$:

(a) $f(x) = 3x$

(b) $f(x) = x^2$

(c) $f(x) = x^2 - 4x + 3$

(d) $f(x) = x^3 - 9x^2 + 15x - 5$

(e) $f(x) = e^x - x^2$

(f) $f(x) = \ln(x) - x$

Exercise 3.2. Determine for which intervals $f(x) = -\frac{1}{3}x^3 + 2x^2 - 3x + 1$ is increasing and decreasing.

3.2 Locating Maxima and Minima

Exercise 3.3. Differentiate the following functions, and thus locate all stationary points:

(a) $f(x) = x^2$

(b) $f(x) = \ln(x) - x$

(c) $f(x) = x^2 - x$

(d) $f(x) = x^3 + 3x^2 - 9x + 2$

(e) $f(x) = g(x^2 - x)$ where g is a function with stationary points at $x = 2$.

3.3 Classifying Stationary Points

Exercise 3.4. Find $f'(x)$ and $f''(x)$ when $f(x) = 2x^5 - 3x^3 + 2x$.

Exercise 3.5. Find all stationary points of the following functions, and apply the second-order condition to determine whether it is a maximum or a minimum:

(a) $f(x) = x^2$

(b) $f(x) = -(x+4)^2$

(c) $f(x) = 2x^3 - 15x^2 - 84x + 108$

(d) $f(x) = 3e^x - \dfrac{1}{2}e^3x^2$