FORK1005
Preparatory Course in Mathematics 2015/16
Lecture 3

July 29, 2015

1 Introduction to Differentiation

To differentiate is to calculate the derivative of a function. A function's derivative is the rate of change of the function for each point x. This builds on the idea of a slope, which we studied for linear functions $f(x) = mx + c$.

1.1 Slope

For any function f, we could define the slope between two points x_1 and x_2 by

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Recall for example that for the linear function $f(x) = mx + c$, the slope between any two points is always m. Another way of expressing the slope between two points is

$$\frac{\Delta f}{\Delta x}.$$

Here, the symbol Δ means *change in*, so $\Delta f = f(x_2) - f(x_1)$ and $\Delta x = x_2 - x_1$.

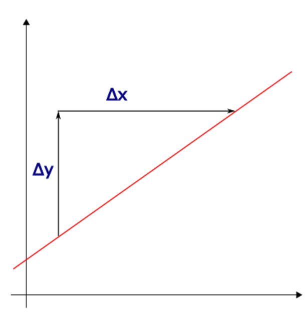


Figure 1.1: The slope is the change in y divided by the change in x

We can also look at the slope between two points for a non-linear graph:

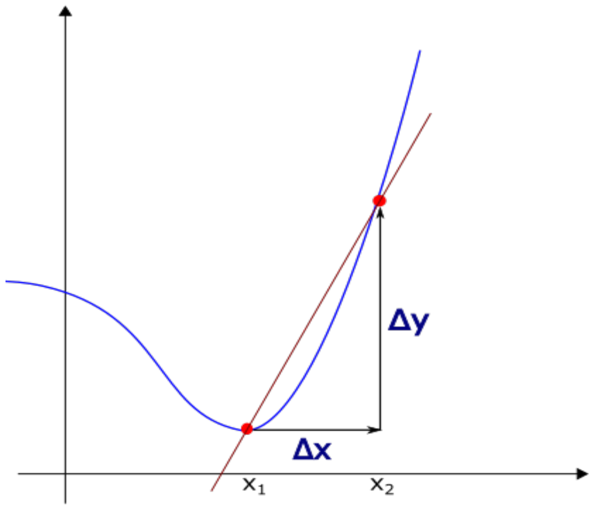


Figure 1.2: The slope between two points for non-linear graph

The slope in the above graph tells us how much the function f has risen from x_1 to x_2. But now the slope changes depending on x_1 and x_2, since the graph is curved. It no longer makes sense to talk about the slope of a curve, because different points give different slopes.

1.2 Tangent

Instead, we think of each point x as having its own slope. To determine the slope of f at a point x, we have to draw the tangent line of f at the point x.

Definition 1.1 (Tangent)**.** Let f be a function. The tangent line to f at a point x is the unique straight line that passes through the coordinate $(x, f(x))$ and that is "going in the same direction" as f at x.

Another way of saying this is that the tangent line to f at x is the straight line that "just touches" the function curve at x.

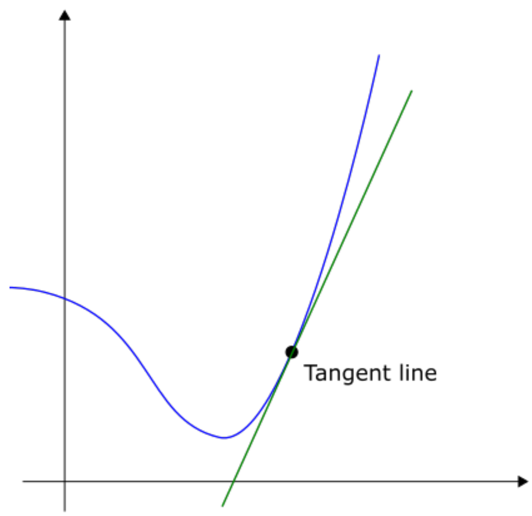


Figure 1.3: The tangent (or slope) of the function at a point x,

Once we have defined the tangent to f at x, we can also define the slope, or the rate of change, of f at a point x:

Definition 1.2 (Rate of change)**.** Given a function f and a point x, the rate of change of f at x is the slope of the tangent line to f at x.

Intuitively, the rate of change of f at x is the rate of change of f **at that specific moment**. Move x slightly, and the rate of change might increase or decrease.

We conclude this section by saying that the derivative of f at x is exactly the rate of change of f at x, and when we talk about differentiating a function, we are just talking about evaluating the slope of the function at each point x:

Definition 1.3 (The derivative of f at x)**.** The derivative of f at x is the rate of change of f at x. In other words, it is the slope of the tangent line to f at x. We denote the derivative of f at x by $f'(x)$ or $\dfrac{\mathrm{d}f}{\mathrm{d}x}(x)$.

1.3 Motivation

Why do we care about rates of change and differentiation?

1. Rates of change is of great interest in all sciences. For example: Inflation in economics is the rate of change of the price level in a country.
2. Rates of change help us describe the world: The speed of a car is the rate of change of the car's position. The acceleration of a car is the rate of change of it's speed. Derivatives tell us in what direction something is changing, and how quickly.
3. Differentiation can be used for optimization: Suppose a company's profits is given by a profit function $P(x)$, where x is the output level of its product. Then if the company wanted to maximize its profits, it could look at the derivative of $P(x)$ to find out where the function reaches its maximum.

2 Differentiation

Hopefully you now get the idea of differentiating, and why it's important. Now we'll move on to the formal mathematics of differentiation.

2.1 Formal Definition of Differentiation

Recall that the derivative of f at x, or $f'(x)$, is the slope of the tangent line to f at x. So how do we calcuate this slope?

1. Let $h > 0$ be a small positive number.
2. Recall that the slope between the points $f(x)$ and $f(x+h)$ is given by
$$\frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}.$$
3. When h is very small, the slope between $f(x)$ and $f(x+h)$ gets very close to the tangent line to f at x.
4. So if we let h ”go to zero”, the slope value
$$\frac{f(x+h)-f(x)}{h}$$
goes to the derivative $f'(x)$.

To illustrate the point of decreasing h, see the figures below:

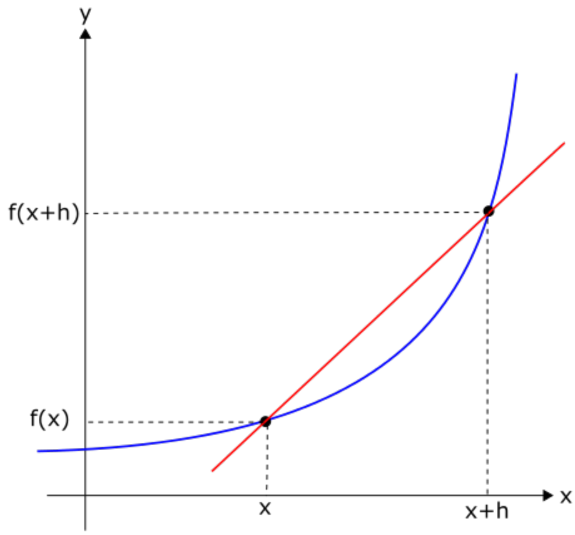


Figure 2.1: Big h

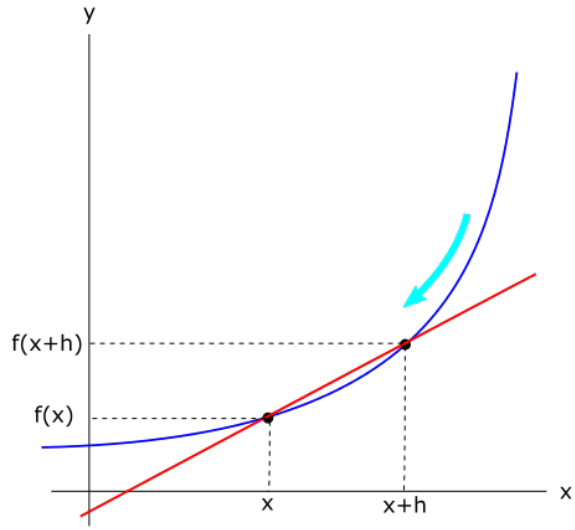


Figure 2.2: Smaller h

Figure 2.3: As h goes towards zero, the slope approaches the tangent to f at x.

Definition 2.1 (Derivative)**.** Let f be a function and x a point. The derivative of f at x is given by the formula

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

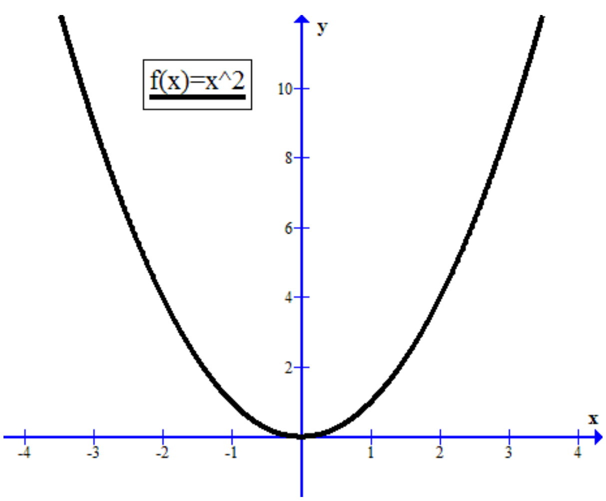


Figure 2.4: Graph of $f(x) = x^2$

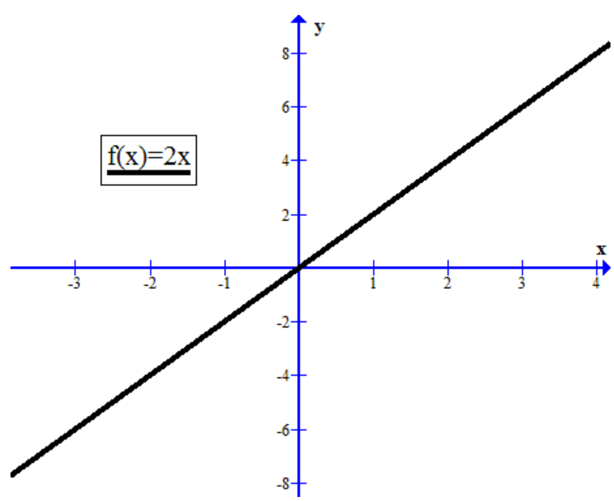


Figure 2.5: Graph of derivative, $f'(x) = 2x$

Above, the left figure shows the graph of $f(x) = x^2$, while the right figure shows its derivative, which is $f'(x) = 2x$. Notice that when $f(x)$ is decreasing, $f'(x)$ is negative, and when $f(x)$ is increasing, $f'(x)$ is positive.

2.2 Examples of Derivatives

Let's calculate some derivatives.

Textbook recipe for calculating derivative:

1. Add $h > 0$ to x and compute $f(x+h)$.
2. Compute the change in f, $f(x+h) - f(x)$.
3. Write out the slope quotient
$$\frac{f(x+h) - f(x)}{h}.$$
4. Simplify the fraction, trying to cancel out as many h-terms as possible.
5. Then $f'(x)$ is the limit of $\dfrac{f(x+h) - f(x)}{h}$ as h goes to 0.

Example 2.2 (Linear functions)**.** The derivative of a linear function is an easy one, because the slope is the same everywhere. Take

$$f(x) = 5x - 2.$$

For any $h > 0$, we have

$$\frac{f(x+h) - f(x)}{h} = \frac{(5(x+h) - 2) - (5x - 2)}{h} = \frac{5h}{h} = 5.$$

Since the slope remains constant for all values of h, we can conclude that the limit of

$$\frac{f(x+h) - f(x)}{h}$$

as h goes to zero also equals 5. That is,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} 5 = 5,$$

for all x.

So the derivative of $f(x) = 5x - 2$ is constant and is equal to 5, which is what we'd expect, as f is a linear function with slope 5.

Example 2.3 (The square function)**.** Consider $f(x) = x^2$. You can see the curve of this function in the figure above. For any $h > 0$, we have

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h} = 2x + h.$$

We take h to zero, and get the derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}(2x + h) = \lim_{h\to 0} 2x + \lim_{h\to 0} h = 2x + 0 = 2x.$$

2.3 Table of Derivatives and Rules

Table of Derivatives:

$\mathbf{f(x)}$	$\mathbf{f'(x)}$
x	1
x^n	nx^{n-1}
a^x	$\ln(a) \cdot a^x$
e^x	$\ln(e) \cdot e^x = e^x$
$\log_a(x)$	$\dfrac{1}{\ln(a)} \cdot \dfrac{1}{x}$

Table of Rules:

Addition	$F(x) = f(x) + g(x) \implies F'(x) = f'(x) + g'(x)$
Coefficient	$F(x) = cf(x) \implies F'(x) = cf'(x)$
Product rule	$F(x) = f(x) \cdot g(x) \implies F'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
Quotient rule	$F(x) = \dfrac{f(x)}{g(x)} \implies F'(x) = \dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{\big(g(x)\big)^2}$
Chain rule	$F(x) = f\left(g(x)\right) \implies F'(x) = g'(x) \cdot f'\left(g(x)\right)$

More concisely:

Addition	$(f+g)' = f' + g'$
Coefficient	$(cf)' = cf'$
Product rule	$(fg)' = f'g + fg'$
Quotient rule	$\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$
Chain rule	$\big(f(g)\big)' = g'f'(g)$

A possibly easier way of remembering the chain rule: Suppose that we have $y = f(u)$ and $u = g(x)$. Then $y = f(g(x))$ is a composite function of x, and the derivative of y with respect to x is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

3 Applications of the Derivative

3.1 Knowing When a Function is Increasing/Decreasing

We know the following facts about the derivative of a function:

1. **$f(x)$ is increasing if and only if $f'(x) > 0$.**
2. **$f(x)$ is decreasing if and only if $f'(x) < 0$.**

This makes it easier to see on what intervals certain functions are increasing or decreasing.

Example 3.1. Consider the function $f(x) = 2\sqrt{x-1} - \dfrac{x}{2} + 1$. We want to know where this function is increasing:

1. We differentiate f:

$$\begin{aligned} f'(x) &= 2 \cdot \frac{1}{2}(x-1)^{1/2-1} - \frac{1}{2} \\ &= \frac{1}{\sqrt{x-1}} - \frac{1}{2}. \end{aligned}$$

2. f is increasing whenever $f'(x) > 0$, so we want

$$f'(x) = \frac{1}{\sqrt{x-1}} - \frac{1}{2} > 0$$

and rearranging

$$\begin{aligned} \frac{1}{\sqrt{x-1}} &> \frac{1}{2} \\ 2 &> \sqrt{x-1} \\ 2^2 &> \left(\sqrt{x-1}\right)^2 \\ 4 &> x-1 \\ 5 &> x. \end{aligned}$$

3. So $f'(x)$ is positive whenever $x < 5$. Equivalently, f is increasing as long as $x < 5$.

4. Since $\sqrt{x-1}$ is only defined when $x - 1 \geq 0$, our function is only defined for $x \geq 1$. We therefore conclude that f is increasing for $1 \geq x < 5$.

5. Similarly, we see that $f' < 0$ whenever $x > 5$, so f is decreasing for $x > 5$.

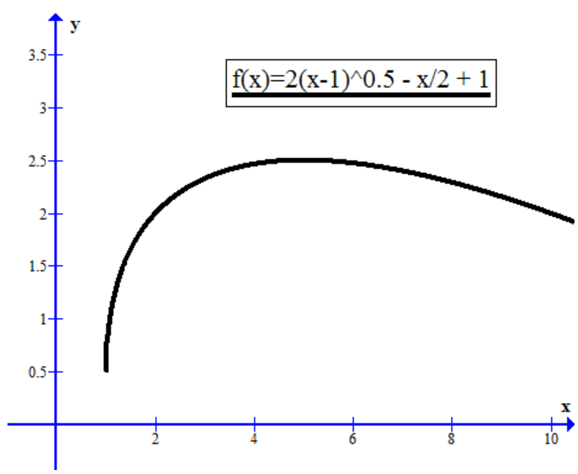


Figure 3.1: The function increase up to 5 and then decreases

3.2 Locating Maxima and Minima

Determining where a function f is maximized and/or minimized is one of the most important applications of derivatives.

Definition 3.2 (Maximum)**.** Let f be a function. We say that x_* maximizes f if

$$f(x_*) \geq f(x)$$

for all x.

Definition 3.3 (Minimum)**.** Let f be a function. We say that x_* minimizes f if

$$f(x_*) \leq f(x)$$

for all x.

Definition 3.4 (Stationary Point)**.** Let f be a function. A stationary point of f is a point x_* such that

$$f'(x_*) = 0.$$

Important fact: If x_* is a maximizer or minimizer of a function f, then

$$\boldsymbol{f'(x_*) = 0.}$$

In other words, all maximizers and minimizers are stationary points. Why is this?

- The simplest explanation is that at a maximum or a minimum, the tanget to the graph is a horizontal line, and a horizontal line has a slope of zero!
- A more technical explanation: If x_* is a maximizer, then $f(x)$ is increasing for $x < x_*$ and decreasing for $x > x_*$. In other words, $f'(x)$ positive when $x < x_*$ and negative when $x > x_*$. Therefore, the graph of f' must be crossing the x-axis at x_*, which means that $f'(x_*) = 0$.
- And if x_* is a minimizer, then we have the reverse situation: $f(x)$ is decreasing for $x < x_*$ and increasing for $x > x_*$, so $f'(x)$ is negative when $x < x_*$ and positive when $x > x_*$. Again, the graph of f' has to cross the x-axis at x_*, so $f'(x_*) = 0$.

Example 3.5. The function below has a maximum at $x = 1$. Therefore, for $x < 1$, f is increasing and $f'(x) > 0$. And for $x > 1$, f is decreasing and $f'(x) < 0$. Therefore $f'(1) = 0$.

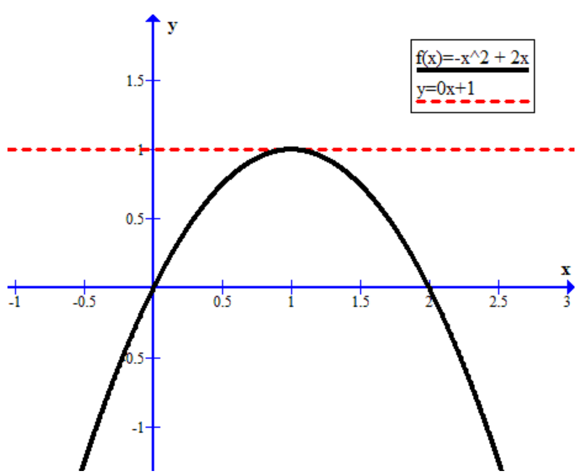


Figure 3.2: The function has a maximum at $x = 1$.

How can we utilize the fact that $f'(x) = 0$ for all minimizers and maximizers? We can use it to locate maximums and minimums:

Example 3.6. A company produces one product (televisons) and it has to decide on how many televisions to produce this year. The company knows that if it produces x televisions this year, its profit will be

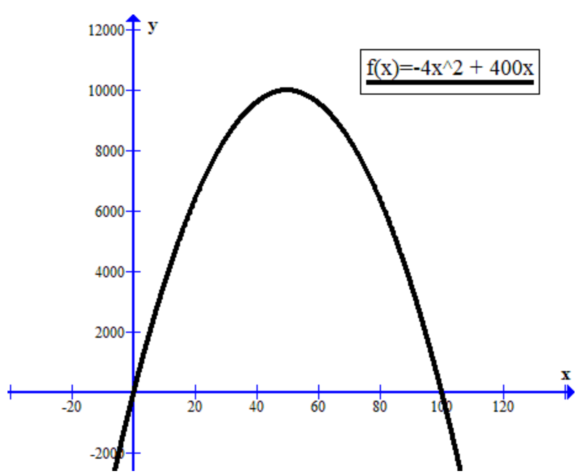
$$P(x) = -4x^2 + 400x.$$

Since the company wants to maximize profits, it wants to maximize $P(x)$ with respect to x.

1. We differentiate P:
$$P'(x) = -4 \cdot 2x + 400 = 400 - 8x.$$

2. We know that any maximizer (or minimizer) x_* will satisfy $P'(x_*) = 0$. So we set up the equality
$$\begin{aligned} 0 &= P'(x) = 400 - 8x \\ 8x &= 400 \\ x &= 50. \end{aligned}$$

3. So $x = 50$ is the only point which satisfies $P'(x)$. And knowing the shape of a quadratic function of the form $P(x) = -4x^2 + 400x$, as displayed in the graph below, we can deduce that $x = 50$ must be a maximizer.



4. So $x_* := 50$ maximizes the company's profits, which is
$$P(100) = -4 \cdot 50^2 + 400 \cdot 50 = 20,000 - 10,000 = 10,000.$$

3.3 Classifying Stationary Points

In the last section, we saw that minimizers and maximizers of f are also stationary points of f: $f'(x) = 0$. Therefore, locating the stationary points of f can help us in locating a maximizer or a minimizer of f. **However, there are three types of stationary points:**

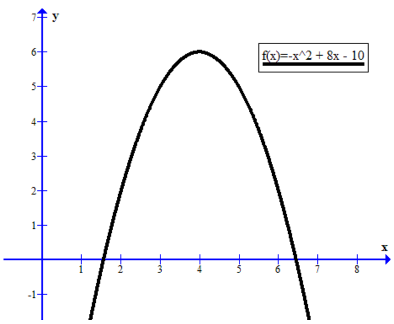


Figure 3.3: Maximum

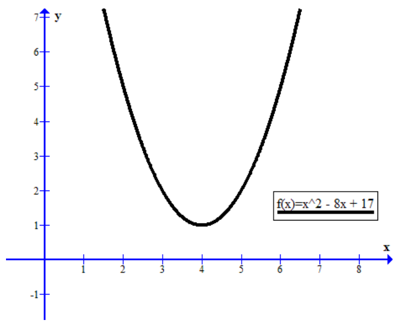


Figure 3.4: Minimum

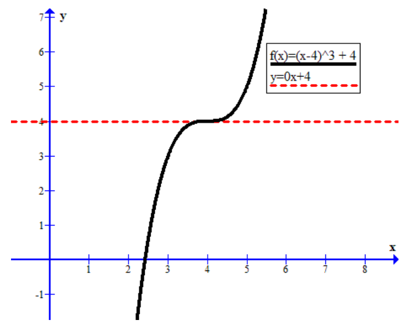


Figure 3.5: Inflection

So how do we know when a stationary point is a maximizer or a minimizer? In the example of maximizing profits above, we cheated a bit, since we knew what the graph looked like. But often, we want to maximize more complicated graphs and we need a mathematically sound way of checking if a stationary point is a maximizer, minimizer or a point of inflection. Thankfully, **this can easily be done by taking the derivative of the derivative.**

When differentiating a function f, we get a new function f'. We can also differentiate this function, to get a *derivative of a derivative*: The derivative of f' is denoted f'' or $\dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2}$.

Definition 3.7 (First- and Second-Order Derivative)**.** Let f be a function. Then f' is called its *first-order derivative* and f'' its *second-order derivative.*

Example 3.8. Suppose a car is driving along a straight line. We let $p(t)$ denote the position of the car at time t. Then, by differentiating p, we get a new function $p'(t)$ which is the car's speed at time t. Then, differentiating p', we get another function $p''(t)$ which is the car's acceleration at time t.

We have the following, very useful result:

Theorem 3.9 (Second-order condition)**.** *Suppose f is a function and x_* is a stationary point, so a point that satisfies $f'(x_*) = 0$. Then if*

1. *$f''(x_*) < 0$, x_* is a maximizer.*
2. *$f''(x_*) > 0$, x_* is a minimizer.*
3. *$f''(x_*) = 0$, it could be a minimizer, maximizer or a point of inflection.*

Sketch proof.

1. If x_* is a maximizer, then $f(x)$ is increasing for $x < x_*$ and decreasing for $x > x_*$. In other words, $f'(x) > 0$ for $x < x_*$ and $f'(x) < 0$ for $x > x_*$. This means than f' is decreasing at x_*, since when $x < x_*$, $f'(x)$ is positive and when $x > x_*$, $f'(x)$ is negative. Therefore $f''(x_*) < 0$.

2. If x_* is a minimizer, then $f(x)$ is decreasing for $x < x_*$ and increasing for $x > x_*$. In other words, $f'(x) < 0$ for $x < x_*$ and $f'(x) > 0$ for $x > x_*$. This means than f' is increasing at x_*, since when $x < x_*$, $f'(x)$ is negative and when $x > x_*$, $f'(x)$ is positive. Therefore $f''(x_*) > 0$.

3. If $f''(x_*) = 0$, then we can't say anything. The functions x^3, x^4 and $-x^4$ all satisfy $f'(0) = f''(0) = 0$, but the first function has a point of inflection, the second has a minimizer and the third has a maximizer.

$\square$