FORK1003
Preparatory Course in Linear Algebra 2016/17
Lecture 3: Determinants

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1 Introduction to Determinants

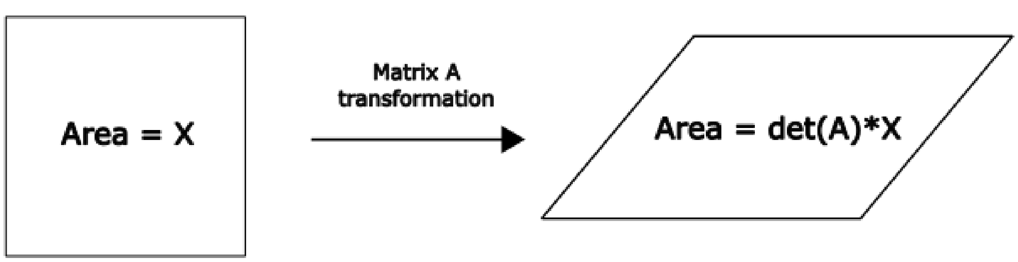
1.1 Understanding the Determinant

Every square ($n \times n$) matrix has a *determinant*, which is a number. For a matrix A, we denote its determinant by $\det(A)$ and/or $|A|$:

$$\textit{(matrix)} \qquad A \quad \longrightarrow \quad \det(A) \qquad \textit{(number)}.$$

It is difficult to explain what the determinant actually *represents*; it is easier to use it than it is to understand it. One explanation of the determinant is the following:

An $n \times n$ matrix is a transformation of n-dimensional geometric shapes. For example, applying a matrix to an n-dimensional square, will transform it to an n-dimensional paralellogram:



The area of the transformed parallelogram is equal to the area of the square times the determinant.

The determinant gives us very useful information about the matrix. Perhaps the most useful piece of information it provides is the following:

Theorem 1.1. *A square matrix A is invertible if and only if*

$$\det(A) \neq 0.$$

1.2 Defining the Determinant

In the last lecture we gave the formula for the determinant of a 2×2 matrix:

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \text{then } \det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

The general definition of a determinant is quite complicated and mathematical, and you don't need to understand it or know it. But for those who are curiuos, this is how you define the determinant of a general $n \times n$ matrix:

Definition 1.2 (Non-compulsory)**.** Let $A = (a_{ij})$ be a $n \times n$ matrix. Then its determinant $\det(A)$ is given by

$$\det(A) := \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \left(\prod_{i=1}^{n} a_{i\sigma(i)} \right)$$

Another obstacle in dealing with determinants is that they become very difficult to calculate when your matrix is 4×4 or bigger:

- When your matrix is 2×2, you only need to add two terms ($a_{11}a_{22}$ and $-a_{12}a_{21}$) to get the determinant.
- For a 3×3 matrix, you add together six terms.
- For a 4×4 matrix, you add together 24 terms.
- ...For a $n \times n$ matrix, you add together $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$ terms!

So we stick to knowing how to calculate the determinant for 2×2 and 3×3 matrices, and for larger matrices, we will use other tools and tricks to find the determinant.

2 Clever Trick for 3×3 Determinants

Before we delve into the theory of determinants, we will show two neat tricks for calculating 2×2 and 3×3 determinants:

Suppose you have the 3×3 matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 4 & 3 & 7 \\ -2 & 5 & 0 \end{bmatrix}$$

1. Write out the matrix again, but add column 1 and 2 to the right of the matrix:

$$\begin{array}{rrr|rr} 1 & 0 & 2 & 1 & 0 \\ 4 & 3 & 7 & 4 & 3 \\ -2 & 5 & 0 & -2 & 5 \end{array}$$

2. Draw 6 diagonal arrows: 3 going to the left and 3 going to the right:

$$\begin{array}{rrr|rr} + & + & + \ - & - & - \\ 1 & 0 & 2 & 1 & 0 \\ 4 & 3 & 7 & 4 & 3 \\ -2 & 5 & 0 & -2 & 5 \end{array}$$

3. For each diagonal line, multiply together the three numbers it crosses. Then you get the determinant of A by adding all the products of the green lines, and subtracting all the products of the red lines:

$$\begin{aligned} 1 \cdot 3 \cdot 0 &= 0, & 0 \cdot 7 \cdot (-2) &= 0, & 2 \cdot 4 \cdot 5 &= 40 \\ 2 \cdot 3 \cdot (-2) &= -12, & 1 \cdot 7 \cdot 5 &= 35, & 0 \cdot 4 \cdot 0 &= 0 \end{aligned}$$

so

$$\begin{aligned} \det(A) &= 0 + 0 + 40 - (-12) - 35 - 0 \\ &= 40 + 12 - 35 \\ &= 17. \end{aligned}$$

Warning: This method does not work for matrices larger than 3×3.

3 Cofactor Expansion

Cofactor expansion is the first method we will use to calculate determinants. The idea of cofactor expansion is that we can express the determinant of a large matrix as the sum of the determinants of many smaller matrices. For example, if we want to find the determinant of a 5×5 matrix, instead of calculating the determinant directly, we can separate the 5×5 matrix into five 4×4 matrices, calculate the determinant of these matrices, and add them together to get the determinant of the 5×5 matrix.

3.1 Minors and Cofactors

If you have a square matrix A, you can obtain a smaller square matrix by removing a row and a column from matrix A.

Example 3.1. Consider the 3×3 matrix

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -7 & 5 \\ 1 & -1 & 6 \end{bmatrix}$$

and denote by A_{ij} the 2×2 matrix you get by removing the ith row and the jth column of matrix A. So for A_{13} you remove the 1st row and the 3rd column:

$$A_{13} = \begin{bmatrix} 2 & -7 \\ 1 & -1 \end{bmatrix}.$$

Similarly, you have

$$A_{22} = \begin{bmatrix} 3 & 1 \\ 1 & 6 \end{bmatrix}, \qquad A_{32} = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}, \qquad A_{11} = \begin{bmatrix} -7 & 5 \\ -1 & 6 \end{bmatrix}.$$

We use the same notation for a general $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

A_{25} is for example the $(n-1) \times (n-1)$ matrix you get by removing the 2nd row and 5th column from A.

These submatrices have determinants, and we call them *minors*:

Definition 3.2 (Minor)**.** Consider a $n \times n$ matrix $A = (a_{ij})$. The *minor* M_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix A_{ij},

$$M_{ij} = \det(A_{ij}).$$

And cofactors are the same as minors, just that you multiply the determinant by a factor of $(-1)^{i+j}$:

Definition 3.3 (Cofactor)**.** Consider a $n \times n$ matrix $A = (a_{ij})$. The *cofactor* C_{ij} is given by

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

Remark 3.4. Note that

$$(-1)^k = \begin{cases} 1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd} \end{cases}$$

So $C_{ij} = M_{ij}$ whenever the row (i) and column (j) add up to an even number $i+j$, and $C_{ij} = -M_{ij}$ whenever $i+j$ is odd. You can also determine the sign of C_{ij} from the (i,j) position in this "sign matrix":

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example 3.5. Consider again the matrix

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -7 & 5 \\ 1 & -1 & 6 \end{bmatrix}.$$

We have

$$\begin{aligned} A_{11} &= \begin{bmatrix} -7 & 5 \\ -1 & 6 \end{bmatrix}, \\ M_{11} &= \det(A_{11}) = -7 \cdot 6 - 5 \cdot (-1) = -42 + 5 = -37, \\ C_{11} &= (-1)^{1+1} M_{11} = M_{11} = -37. \end{aligned}$$

Similarly, we have

$$\begin{aligned} A_{32} &= \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}, \\ M_{32} &= \det(A_{32}) = 3 \cdot 5 - 1 \cdot 2 = 13, \\ C_{32} &= (-1)^{3+2} M_{32} = -M_{32} = -13. \end{aligned}$$

3.2 Cofactor Expansion

Suppose you have a $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}.$$

Then the determinant $\det(A)$ can be found by calculating all the cofactors along a row or a column:

Theorem 3.6 (Cofactor expansion along a row)**.** *Pick a row k. Then*

$$\det(A) = a_{k1}C_{k1} + a_{k2}C_{k2} + \dots a_{kn}C_{kn} = \sum_{j=1}^{n} a_{kj}C_{kj}.$$

Theorem 3.7 (Cofactor expansion along a column)**.** *Pick a column k. Then*

$$\det(A) = a_{1k}C_{1k} + a_{2k}C_{2k} + \dots a_{nk}C_{nk} = \sum_{j=1}^{n} a_{jk}C_{jk}.$$

Remark 3.8. The most common way is to do row expansion along the 1st row, unless there is another more obvious option.

Example 3.9. Consider the 3×3 matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 1 & 4 \\ 2 & -1 & 3 \end{bmatrix}$$

We calculate the determinant by cofactor expanding the 1st row:

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & 2 & 0 \\ -3 & 1 & 4 \\ 2 & -1 & 3 \end{vmatrix} \\ &= 1 \cdot \begin{vmatrix} 1 & 4 \\ -1 & 3 \end{vmatrix} - 2 \cdot \begin{vmatrix} -3 & 4 \\ 2 & 3 \end{vmatrix} + 0 \cdot \begin{vmatrix} -3 & 1 \\ 2 & -1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 4 \\ -1 & 3 \end{vmatrix} - 2 \begin{vmatrix} -3 & 4 \\ 2 & 3 \end{vmatrix} \\ &= \big(1 \cdot 3 - (-1) \cdot 4\big) - 2\,(-3 \cdot 3 - 2 \cdot 4) \\ &= 41. \end{aligned}$$

Example 3.10. We want to calculate the determinant of

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

The 2nd column has two zeros, so two of the terms in the expansion will disappear. So for this matrix, it is better to expand along column 2:

$$\begin{aligned} \begin{vmatrix} 1 & 2 & -3 \\ 3 & 0 & 1 \\ 2 & 0 & -1 \end{vmatrix} &= -2 \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} + 0 \begin{vmatrix} 1 & -3 \\ 2 & -1 \end{vmatrix} - 0 \begin{vmatrix} 1 & -3 \\ 3 & 1 \end{vmatrix} \\ &= -2 \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} \\ &= -2(-3-2) \\ &= 10. \end{aligned}$$

Example 3.11. Lastly, let's calculate the determinant of the following 4×4 matrix:

$$A = \begin{bmatrix} 1 & 6 & -3 & 13 \\ 0 & 2 & -3 & 8 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 2 & -1 \end{bmatrix}$$

The first column has 3 zeros so let's expand along that:

$$\begin{aligned} \begin{vmatrix} 1 & 6 & -3 & 13 \\ 0 & 2 & -3 & 8 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 2 & -1 \end{vmatrix} &= 1 \cdot \begin{vmatrix} 2 & -3 & 8 \\ 0 & 3 & -2 \\ 0 & 2 & -1 \end{vmatrix} - 0 + 0 - 0 \\ &= \begin{vmatrix} 2 & -3 & 8 \\ 0 & 3 & -2 \\ 0 & 2 & -1 \end{vmatrix}. \end{aligned}$$

This 3×3 determinant has 2 zeros in the first column, so we can expand again:

$$\begin{aligned} \begin{vmatrix} 2 & -3 & 8 \\ 0 & 3 & -2 \\ 0 & 2 & -1 \end{vmatrix} &= 2 \cdot \begin{vmatrix} 3 & -2 \\ 2 & -1 \end{vmatrix} \\ &= 2(-3+4) = 2. \end{aligned}$$

So $\det(A) = 2$.

4 Determinants by Row Reduction

In this section, we will look at how we can simplify the determinant calculations by row reducing the matrix. Once again, we are using elementary row operations. The first thing we need to cover is how applying elementary row operations affect the determinant.

4.1 Determinants and Elementary Row Operations

We have three elementary row operations:

1. Scalar multiplication: Multiplying a row by a nonzero constant: $(R1 \to cR1)$.
2. Row addition: Adding a multiple of a row to another row: $(R2 \to R2 + 5R1)$.
3. Interchanging: Swapping two rows: $(R2 \leftrightarrow R3)$.

Theorem 4.1. *Suppose A is an $n \times n$ matrix with determinant $|A|$.*

1. *When multiplying a row by a scalar c, multiply the determinant by the same scalar c:*
   If B is the matrix we get from A by replacing $R1$ by $cR1$, then
   $$|B| = c|A|.$$
2. *When adding a multiple of another row to a row, the determinant stays the same:*
   If B is the matrix we get from A by replacing $R2$ by $R2 + cR3$, then
   $$|B| = |A|.$$
3. *When swapping two rows, multiply the determinant by (-1):*
   If B is the matrix we get from A by interchanging $R1$ and $R3$, then
   $$|B| = -|A|.$$

So in conclusion, it is fairly straightforward to row reduce a matrix and still keep track of what its determinant will be.

Example 4.2. Suppose we know that the determinant of

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -5 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

is

$$\det(A) = -5.$$

Then

1. Scalar multiplication:

$$\begin{vmatrix} 1 & 0 & 3 \\ 8 & -20 & 4 \\ 0 & 1 & 2 \end{vmatrix} = -20, \qquad R2 \to 4R2$$

2. Row addition:

$$\begin{vmatrix} 1 & 0 & 3 \\ 2 & -5 & 1 \\ 3 & 1 & 11 \end{vmatrix} = -5. \qquad R3 \to R3 + 3R1$$

3. Row swapping:

$$\begin{vmatrix} 2 & -5 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & 2 \end{vmatrix} = 5, \qquad R1 \leftrightarrow R2$$

4.2 Determinant of Upper-Diagonal Matrices

In the last section, we saw that when row reducing a matrix, you can write the determinant of your original matrix in terms of the determinant of the row reduced matrix. In this section, we will take advantage of this to reduce matrices to *upper-diagonal* matrices.

Definition 4.3. A upper-diagonal matrix is any $n \times n$ matrix where all entries below the diagonal are zero:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

Proposition 4.4. *Suppose A is $n \times n$ and upper-diagonal as above. Then*

$$\det(A) = a_{11} \cdot a_{22} \cdot a_{33} \cdot \ldots \cdot a_{nn}.$$

Idea of proof. Expand along the first column to get

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} & \ldots & a_{2n} \\ 0 & a_{33} & \ldots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{nn} \end{vmatrix}$$

Then expand this along its first column to get

$$\det(A) = a_{11}a_{22} \begin{vmatrix} a_{33} & a_{34} & \ldots & a_{3n} \\ 0 & a_{44} & \ldots & a_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{nn} \end{vmatrix}$$

and so on until you get

$$\det(A) = a_{11} \cdot a_{22} \cdot a_{33} \cdot \ldots \cdot a_{nn}.$$

□

Example 4.5. The determinant of

$$A = \begin{bmatrix} 1 & 4 & -24 & 64 \\ 0 & -2 & 35 & -19 \\ 0 & 0 & 3 & 97 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

is

$$\det(A) = 1 \cdot (-2) \cdot 3 \cdot (-3) = 18.$$

Example 4.6. The identity matrix I_n has determinant $\det(I_n) = 1$.

So if we row reduce a matrix to an upper-diagonal matrix, we can easily calculate the determinant.

Example 4.7. Using row reduction, calculate the determinant of the following matrix:

$$A = \begin{bmatrix} 2 & 8 & 4 & -6 \\ -2 & -5 & 2 & 11 \\ -1 & 2 & 3 & 13 \\ 1 & 4 & 2 & -5 \end{bmatrix}$$

In the following row reduction, the matrix will be on the left side, the row operation in the middle, and the new determinant in terms of $|A|$ on the right side:

$$\begin{bmatrix} 2 & 8 & 4 & -6 \\ -2 & -5 & 2 & 11 \\ -1 & 2 & 3 & 13 \\ 1 & 4 & 2 & -5 \end{bmatrix} \qquad\qquad |A|$$

$$\begin{bmatrix} 1 & 4 & 2 & -3 \\ -2 & -5 & 2 & 11 \\ -1 & 2 & 3 & 13 \\ 1 & 4 & 2 & -5 \end{bmatrix} \qquad R1 \to \frac{1}{2}R1 \qquad \frac{1}{2}|A|$$

$$\begin{bmatrix} 1 & 4 & 2 & -3 \\ 0 & 3 & 6 & 5 \\ 0 & 6 & 5 & 10 \\ 0 & 0 & 0 & -2 \end{bmatrix} \qquad \begin{matrix} R2 \to R2 + 2R1 \\ R3 \to R3 + R1 \\ R4 \to R4 - R1 \end{matrix} \qquad \frac{1}{2}|A|$$

$$\begin{bmatrix} 1 & 4 & 2 & -3 \\ 0 & 3 & 6 & 5 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \qquad R3 \to R3 - 2R2 \qquad \frac{1}{2}|A|$$

The last matrix is upper diagonal so

$$\begin{vmatrix} 1 & 4 & 2 & -3 \\ 0 & 3 & 6 & 5 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix} = 1 \cdot 3 \cdot (-7) \cdot (-2) = 42 = \frac{1}{2}|A|.$$

Therefore,

$$|A| = 84.$$

4.3 Combining Cofactor Expansion and Row Reduction

Sometimes the best approach is a combination of cofactor expansion and row reduction: Reduce the matrix size through cofactor expansion when you can, and row reduce otherwise.

Example 4.8. Calculate the determinant of

$$A = \begin{bmatrix} 3 & 0 & -4 & 2 \\ -1 & 0 & 3 & 5 \\ 2 & 5 & 1 & -2 \\ -3 & 0 & 2 & 6 \end{bmatrix}$$

through a combination of row reduction and cofactor expansion:

1. The 2nd column has 3 zeros so we begin by expanding along this column:

$$|A| = \begin{vmatrix} 3 & 0 & -4 & 2 \\ -1 & 0 & 3 & 5 \\ 2 & 5 & 1 & -2 \\ -3 & 0 & 2 & 6 \end{vmatrix} = -5 \begin{vmatrix} 3 & -4 & 2 \\ -1 & 3 & 5 \\ -3 & 2 & 6 \end{vmatrix}$$

2. Here we could do cofactor expansion again and calculate three 2×2 determinants, but we will use row reduction for the sake of practice:

$$\begin{aligned}
& -5 \begin{vmatrix} 3 & -4 & 2 \\ -1 & 3 & 5 \\ -3 & 2 & 6 \end{vmatrix} \\
&= -5 \begin{vmatrix} 3 & -4 & 2 \\ -1 & 3 & 5 \\ 0 & -2 & 8 \end{vmatrix} && R3 \to R3 + R1 \\
&= -5 \begin{vmatrix} 0 & 5 & 17 \\ -1 & 3 & 5 \\ 0 & -2 & 8 \end{vmatrix} && R1 \to R1 + 3R2 \\
&= +5 \begin{vmatrix} -1 & 3 & 5 \\ 0 & 5 & 17 \\ 0 & -2 & 8 \end{vmatrix} && R1 \leftrightarrow R2 \\
&= +5 \begin{vmatrix} 5 & 17 \\ -2 & 8 \end{vmatrix} && \text{Expand along column 1} \\
&= -1 \cdot 5(40 + 34) \\
&= -370.
\end{aligned}$$

So $|A| = -370$.

5 The Adjugate Matrix and Inverses

In this section we will study the adjugate of matrix A, denoted by $\mathrm{adj}(A)$, and the important identity

$$A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A),$$

which ties together the concepts of inverse matrices, determinants and cofactors.

5.1 The Cofactor and Adjugate Matrices

Recall that the cofactor C_{ij} of a matrix A is given by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

where M_{ij} is the determinant of the matrix obtained by removing the ith row and jth column of A.

Definition 5.1 (Cofactor matrix)**.** The *cofactor matrix* C of A is the matrix

$$C = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}.$$

So it is the matrix that contains each of the cofactors of A.

Definition 5.2 (Adjugate matrix)**.** The *adjugate matrix* $\mathrm{adj}(A)$ of A is the transpose of the cofactor matrix C:

$$\mathrm{adj}(A) = C^T = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

Theorem 5.3. *Let A be an invertible matrix. Then*

$$A^{-1} = \frac{1}{\det(A)} adj(A).$$

Example 5.4. Find the inverse of

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 1 \\ 5 & -1 & -1 \end{bmatrix}$$

using the above adjugate formula:

1. We begin by calculating the cofactor matrix:

$$\begin{aligned} C &= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \\ &= \begin{bmatrix} \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 5 & -1 \end{vmatrix} \\ -\begin{vmatrix} 3 & -1 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 5 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 5 & -1 \end{vmatrix} \\ \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} \end{bmatrix} \\ &= \begin{bmatrix} -1 & 6 & -11 \\ 4 & 4 & 16 \\ 5 & -2 & -1 \end{bmatrix} \end{aligned}$$

2. This gives us the adjugate matrix

$$\operatorname{adj}(A) = C^T = \begin{bmatrix} -1 & 4 & 5 \\ 6 & 4 & -2 \\ -11 & 16 & -1 \end{bmatrix}$$

3. Lastly, we need the determinant of A. We expand along the 1st row:

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & 3 & -1 \\ 1 & 2 & 1 \\ 5 & -1 & -1 \end{vmatrix} \\ &= 1\begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 2 \\ 5 & -1 \end{vmatrix} \\ &= -1 - 3(-6) + (-1)(-11) \\ &= 28. \end{aligned}$$

4. Now we can plug into the identity

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$$

to get

$$\begin{aligned} A^{-1} &= \frac{1}{28}\begin{bmatrix} -1 & 4 & 5 \\ 6 & 4 & -2 \\ -11 & 16 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -1/28 & 1/7 & 5/28 \\ 3/14 & 1/7 & -1/14 \\ -11/28 & 4/7 & -1/28 \end{bmatrix} \end{aligned}$$

6 Cramer's Rule

Cramer's Rule is a peculiar application of determinants to solve linear systems. It is particularly useful if you only want to find the solution for one variable x_i without having to calculate the entire solution for $x_1, x_2, \ldots, x_n$.

Consider the linear system/matrix equation

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n, \end{aligned}$$

or equivalently

$$A\underline{\mathbf{x}} = \underline{\mathbf{b}}$$

with the coefficient matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix},$$

and the constant vector

$$\underline{\mathbf{b}} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Suppose that A is invertible, so that the linear system has a unique solution for every $\underline{\mathbf{b}}$ and also so that $\det(A) \neq 0$. Also, denote by A_i the matrix obtained from A by replacing column i by the column vector $\underline{\mathbf{b}}$. So for example A_1 is

$$A_1 = \begin{bmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \dots & a_{nn} \end{bmatrix},$$

A_2 is

$$A_2 = \begin{bmatrix} a_{11} & b_1 & \dots & a_{1n} \\ a_{12} & b_2 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & b_n & \dots & a_{nn} \end{bmatrix},$$

and A_n is

$$A_n = \begin{bmatrix} a_{11} & a_{12} & \dots & b_1 \\ a_{21} & a_{22} & \dots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & b_n \end{bmatrix}.$$

Theorem 6.1 (Cramer's Rule)**.** *Consider the linear system*

$$A\underline{\mathbf{x}} = \underline{\mathbf{b}},$$

where A is invertible. Then the unique solution for x_i is given by

$$x_i = \frac{\det(A_i)}{\det(A)},$$

for each $i = 1, 2, \dots, n$.

Example 6.2. Using Cramer's rule, solve the following linear system:

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 4 \\ x_1 \phantom{{}+2x_2} + 2x_3 = 10 \\ -x_1 + 2x_2 + x_3 = -4. \end{cases}$$

1. We have the coefficient matrix

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & 2 \\ -1 & 2 & 1 \end{bmatrix},$$

and the three matrices

$$A_1 = \begin{bmatrix} 4 & 2 & 1 \\ 10 & 0 & 2 \\ -4 & 2 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 10 & 2 \\ -1 & -4 & 1 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 0 & 10 \\ -1 & 2 & -4 \end{bmatrix}.$$

2. We calculate the determinants using expansion along the 1st row:

$$\det(A) = 3\begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = -12 - 6 + 2 = -16$$

$$\det(A_1) = 4\begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 10 & 2 \\ -4 & 1 \end{vmatrix} + 1\begin{vmatrix} 10 & 0 \\ -4 & 2 \end{vmatrix} = 4(-4) - 2(18) + 20 = -32$$

$$\det(A_2) = 3\begin{vmatrix} 10 & 2 \\ -4 & 1 \end{vmatrix} - 4\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 10 \\ -1 & -4 \end{vmatrix} = 3(18) - 4(3) + 1(6) = 48$$

$$\det(A_3) = 3\begin{vmatrix} 0 & 10 \\ 2 & -4 \end{vmatrix} - 2\begin{vmatrix} 1 & 10 \\ -1 & -4 \end{vmatrix} + 4\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} = 3(-20) - 2(6) + 4(2) = -64$$

3. By Cramer's rule, the solution is given by

$$\underline{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \dfrac{\det(A_1)}{\det(A)} \\ \\ \dfrac{\det(A_2)}{\det(A)} \\ \\ \dfrac{\det(A_3)}{\det(A)} \end{bmatrix} = \begin{bmatrix} \dfrac{-32}{-16} \\ \\ \dfrac{48}{-16} \\ \\ \dfrac{-64}{-16} \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}.$$

7 Determinant Properties

Here is a table of some of the most important properties of determinants:

Determinant of identity	$\det(I_n) = 1$
Multiplicative property	$\det(AB) = \det(A)\det(B)$
Determinant of inverse	$\det(A^{-1}) = \dfrac{1}{\det(A)}$
Determinant of transpose	$\det(A^T) = \det(A)$
Scalar multiple ($n \times n$ matrix)	$\det(cA) = c^n \det(A)$
NOT additive	$\det(A + B) \neq \det(A) + \det(B)$

