FORK1003
Preparatory Course in Linear Algebra 2016/17
Lecture 1: Linear Systems

August 1, 2015

1 Introduction to Linear Systems

1.1 Linear Equation

Linear equations are of the form

$$x_1 + 2x_2 = 4 \qquad \text{and} \qquad 3x_1 - 4x_2 + x_3 = -2.$$

Definition 1.1. A *linear equation of n variables* is an equation of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b,$$

where $x_1, x_2, \ldots, x_n$ are the variables, and $a_1, a_2, \ldots, a_n, b$ are fixed constants.

We call them linear equations because their graphs are straight lines, or rather straight planes in n dimensions.

Example 1.2.

Linear equations	**Not** linear equations
$3(x_1 - x_2) = -2$	$x_1x_2 + 3x_3 = -1$
$-4x_1 - 2^{1/3}x_2 = 3$	$4x_1^{1/2} + x_2 = 2$
$x_1 - x_3 = 2x_2$	$(x_1 + x_2)(x_3 - x_4) = 3$

1.2 Linear Systems

Linear systems of equations is a collection of linear equations, such as

$$\begin{cases} 3x_1 + 2x_2 = 0 \\ -x_1 + x_2 = 5 \end{cases} \qquad (2 \times 2 - \text{system})$$

or

$$\begin{cases} 3x_1 + 2x_2 \phantom{{}- 2x_3} = 2 \\ x_1 + x_2 - 2x_3 = 0 \\ - x_2 + 4x_3 = -2. \end{cases} \qquad (3 \times 3 - \text{system})$$

Definition 1.3 (Linear System)**.** In general, an $m \times n$-system is a system of m equations in n variables written in the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m. \end{aligned}$$

2 Solutions of Linear Systems

Substitution is the most basic and intuitive way of solving linear systems.

Example 2.1. We will solve the following linear system by substitution:

$$\begin{cases} 2x_1 + x_2 = 3 \\ x_1 - x_2 = 2. \end{cases}$$

We rearrange the second equation to get x_2 in terms of x_1:

$$x_2 = x_1 - 2.$$

Then we plug in this expression for x_2 in the first equation:

$$\begin{aligned} 2x_1 + \underbrace{x_2}_{=x_1-2} &= 3 \\ 2x_1 + x_1 - 2 &= 3 \\ 3x_1 &= 5 \\ x_1 &= 5/3. \end{aligned}$$

So $x_1 = 5/3$, and to solve for x_2, we can plug $x_1 = 5/3$ into either of the equations above:

$$x_2 = \underbrace{x_1}_{=5/3} - 2 = 5/3 - 2 = \frac{5-6}{3} = -1/3.$$

So the solution to this linear system is

$$(x_1, x_2) = (5/3, -1/3).$$

Furthermore, this solution is unique.

Definition 2.2 (Solution)**.** A list of values $(x_1, x_2, \ldots, x_n)$ is a *solution* to a linear system of equations if each equation is true when the values are substituted in.

For a linear system, we are particularly interested in whether a solution exists for the system, and if so, how many solutions exist. The following fact is important:

Proposition 2.3. *A system of linear equation has either*

(i) No solutions (inconsistent),

(ii) One unique solution (consistent), or

(iii) Infinitely many solutions (consistent).

Example 2.4 (Two equations in two variables)**.** The following linear systems have no solutions, one unique solution and infinite solutions respectively, as illustrated by the graphs below.

$$\begin{cases} 3x - y = -1 \\ 3x - y = 2 \end{cases}$$

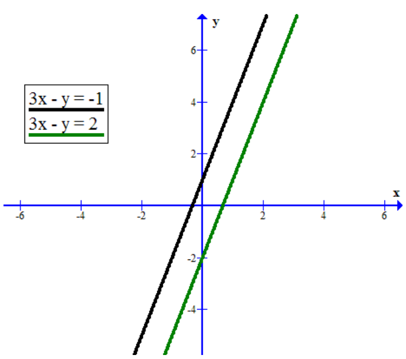

Figure 2.1: No solutions

$$\begin{cases} 2x - y = 4 \\ x + y = 6 \end{cases}$$

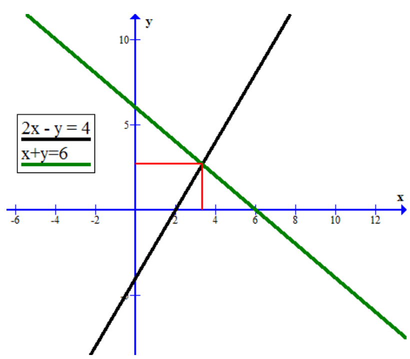

Figure 2.2: One solution

$$\begin{cases} \frac{1}{2}x + 2y = 4 \\ x + 4y = 8 \end{cases}$$

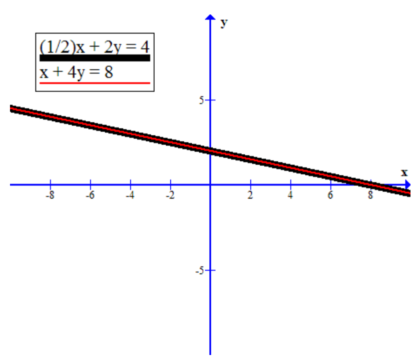

Figure 2.3: Infinite solutions

Example 2.5 (Three equations in three variables)**.** Here are graphs of three 3×3-linear systems. The graph of a linear equation in three variables is a 2D-plane in 3 dimensions. The solutions correspond to the points on the diagram where all graphs intersect simultaneously.

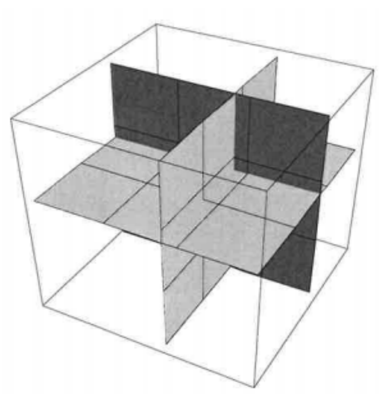

Figure 2.4: One solution

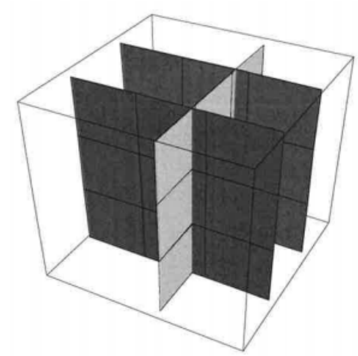

Figure 2.5: No solutions

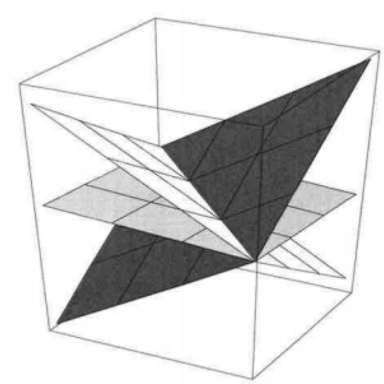

Figure 2.6: Infinite solutions

Figure 2.7: Source: Eivind Eriksen's FK1003 notes, 2014

3 Row Reduction

While the substitution method is sufficient for solving a linear systems with two variables, once you have three or more variables, substitution can become tedious and messy. Using *row reduction* and matrices then becomes the preferred strategy, because it lets you do the calculations while keeping all the equations and variables organized.

3.1 Coefficient & Augmented Matrix

For a linear system, we have two matrices. The *coefficient matrix* has one row for each equation, and one column for each variable, and it contains the coefficients of all the variables. The *augmented matrix* is the same as the coefficient matrix, but has one extra column on the right, which contains the constant terms on the right-hand side of each equation.

$$\begin{cases} -x - 6y = 4 \\ 2x + 3y = 1 \end{cases} \qquad \begin{bmatrix} -1 & -6 \\ 2 & 3 \end{bmatrix} \qquad \left[\begin{array}{cc|c} -1 & -6 & 4 \\ 2 & 3 & 1 \end{array}\right]$$

Linear system **Coefficient matrix** **Augmented matrix**

Definition 3.1 ($m \times n$-matrix)**.** A $m \times n$-matrix is a matrix with m rows and n columns.

So a linear system with m equations in n variables gives you a $m \times n$ coefficient matrix and a $m \times (n + 1)$ augmented matrix.

3.2 Elementary Row Operations

Definition 3.2 (Elementary Row Operations)**.** There are three *elementary row operations* we can use to simplify the augmented matrix of a linear system:

1. (Scaling) Multiply all numbers in a row by a nonzero constant.
2. (Addition) Add a multiple of one row to another.
3. (Interchanging) Interchange two rows.

Example 3.3. We will solve the linear system below by using row operations on its augmented matrix. To the right of each augmented matrix is a description of the row operation used, and on the left is the corresponding linear system for each augmented matrix. Pay particular attention to what happens to the equations for each row operation.

$$\begin{cases} x + y = 3 \\ x - y = -1 \end{cases} \qquad \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & -1 \end{array}\right]$$

$$\begin{cases} 2x \phantom{{}-y} = 2 \\ x - y = -1 \end{cases} \qquad \left[\begin{array}{cc|c} 2 & 0 & 2 \\ 1 & -1 & -1 \end{array}\right] \qquad \begin{array}{c} R1 \to R1 + R2 \\ \text{(Add Row 2 to Row 1)} \end{array}$$

$$\begin{cases} x \phantom{{}-y} = 1 \\ x - y = -1 \end{cases} \qquad \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & -1 & -1 \end{array}\right] \qquad \begin{array}{c} R1 \to \frac{1}{2}R1 \\ \text{(Scale Row 1 by 1/2)} \end{array}$$

$$\begin{cases} x \phantom{{}-y} = 1 \\ -y = -2 \end{cases} \qquad \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & -1 & -2 \end{array}\right] \qquad \begin{array}{c} R2 \to R2 - R1 \\ \text{(Add } -\text{(Row 1) to Row 2)} \end{array}$$

$$\begin{cases} x \phantom{{}-y} = 1 \\ y = 2 \end{cases} \qquad \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \qquad \begin{array}{c} R2 \to -R2 \\ \text{(Scale R2 by } -1) \end{array}$$

Through these row operations, we have arrived at a solution

$$x = 1, \qquad y = 2.$$

Definition 3.4 (Solution set)**.** The solution set of a linear system is the set of all possible solutions.

Definition 3.5 (Equivalent systems)**.** Two linear systems are equivalent if they have the same solution set.

Definition 3.6 (Row equivalent matrices)**.** Two matrices are row equivalent if you can use row operations to convert one matrix into the other.

Very important fact:

Theorem 3.7. *Linear systems are equivalent if and only if their augmented matrices are row equivalent.*

Put differently, row operations preserve solution sets.

Example 3.8. Lets solve a 3×3 linear system using row operations on augmented matrices. Take the linear system

$$\begin{cases} x_1 + 2x_2 - x_3 = 2 \\ 2x_1 - x_2 + 5x_3 = 15 \\ x_1 + 3x_3 = 10. \end{cases}$$

The idea is to use row operations on its augmented matrix, until its coefficient matrix is reduced to the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We start with the augmented matrix below, and show the row operations to the right:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & -1 & 5 & 15 \\ 1 & 0 & 3 & 10 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -5 & 7 & 11 \\ 0 & -2 & 4 & 8 \end{array}\right] \qquad \begin{array}{c} R2 \to R2 - 2R1 \\ R3 \to R3 - R1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -2 & 4 & 8 \\ 0 & -5 & 7 & 11 \end{array}\right] \qquad \begin{array}{c} R2 \leftrightarrow R3 \\ \text{Interchange row 2 and row 3} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -2 & -4 \\ 0 & -5 & 7 & 11 \end{array}\right] \qquad R2 \to \frac{-1}{2}R2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & -3 & -9 \end{array}\right] \qquad \begin{array}{c} R1 \to R1 - 2R2 \\ R3 \to R3 + 5R2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad R3 \to \frac{-1}{3}R3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \begin{array}{l} R1 \to R1 - 3R3 \\ R2 \to R2 + 2R3 \end{array}$$

This augmented matrix represents the linear system

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 3 \end{aligned}$$

which obviously has the unique solution

$$x_1 = 1, \qquad x_2 = 2, \qquad x_3 = 3.$$

Since row equivalent matrices correspond to equivalent linear systems, this is also the solution to the original linear system.
Check: If we plug $(x_1, x_2, x_3) = (1, 2, 3)$ into the original system, we get

$$\begin{cases} x_1 + 2x_2 - x_3 = 2 \\ 2x_1 - x_2 + 5x_3 = 15 \\ x_1 + 3x_3 = 10. \end{cases} \qquad \begin{aligned} 1 + 2 \cdot 2 - 3 &= 1 + 4 - 3 &&= 2 \\ 2 \cdot 1 - 2 + 5 \cdot 3 &= 2 - 2 + 15 &&= 15 \\ 1 + 3 \cdot 3 &= 1 + 9 &&= 10. \end{aligned}$$

3.3 Infinite or no Solutions

In the last section, when solving a linear system that had a unique solution, we were able to row reduce its coefficient matrix to the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The reason why is summarized in the following proposition:

Proposition 3.9. *An $n \times n$ linear system has a unique solution if and only if its coefficient matrix is row equivalent to the identity matrix*

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Now we will see what happens to linear systems with zero or infinitely many solutions:

Example 3.10. Consider the linear system

$$\begin{cases} \qquad\ 3x_2 - \ \ 6x_3 = 9 \\ 6x_1 - 9x_2 + 12x_3 = -1 \\ 5x_1 - 7x_2 + \ \ 9x_3 = 0. \end{cases}$$

We try to row reduce its augmented matrix to the identity matrix:

$$\left[\begin{array}{ccc|c} 0 & 3 & -6 & 9 \\ 6 & -9 & 12 & -1 \\ 5 & -7 & 9 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 6 & -9 & 12 & -1 \\ 0 & 3 & -6 & 9 \\ 5 & -7 & 9 & 0 \end{array}\right] \qquad R1 \leftrightarrow R2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 9 \\ 5 & -7 & 9 & 0 \end{array}\right] \qquad R1 \to R1 - R3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 9 \\ 0 & 3 & -6 & 5 \end{array}\right] \qquad R3 \to R3 - 5R1$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 9 \\ 0 & 0 & 0 & -4 \end{array}\right] \qquad R3 \to R3 - R2$$

This augmented matrix represents the linear system

$$\begin{cases} x_1 - 2x_2 + 3x_3 = -1 \\ \qquad 3x_2 - 6x_3 = 9 \\ 0x_1 + 0x_2 + 0x_3 = -4. \end{cases}$$

Notice that the bottom equation $0 = -4$ is a contradiction with no solutions. Therefore the original system has no solutions and is inconsistent.

Example 3.11. We will now solve the linear system

$$\begin{cases} \qquad\ 3x_2 - \ \ 6x_3 = 9 \\ 6x_1 - 9x_2 + 12x_3 = -1 \\ 5x_1 - 7x_2 + \ \ 9x_3 = 2/3. \end{cases}$$

This is the same linear system as in the last example, except for that the last equation now equals 2/3 instead of 0. We try to row reduce its augmented matrix to the identity matrix:

$$\left[\begin{array}{ccc|c} 0 & 3 & -6 & 9 \\ 6 & -9 & 12 & -1 \\ 5 & -7 & 9 & 2/3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 6 & -9 & 12 & -1 \\ 0 & 3 & -6 & 9 \\ 5 & -7 & 9 & 2/3 \end{array}\right] \qquad R1 \leftrightarrow R2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -5/3 \\ 0 & 3 & -6 & 9 \\ 5 & -7 & 9 & 2/3 \end{array}\right] \qquad R1 \to R1 - R3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -5/3 \\ 0 & 3 & -6 & 9 \\ 0 & 3 & -6 & \underbrace{27/3}_{=9} \end{array}\right] \qquad R3 \to R3 - 5R1$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -5/3 \\ 0 & 3 & -6 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R3 \to R3 - R2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -5/3 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R2 \to \frac{1}{3}R2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 13/3 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R1 \to R1 + 2R2$$

This augmented matrix represents the linear system

$$\begin{cases} x_1 \qquad\quad - \ x_3 = 13/3 \\ \qquad\quad x_2 - 2x_3 = 3 \\ 0x_1 + 0x_2 + 0x_3 = 0. \end{cases}$$

Now the last equation says $0 = 0$, which is always true. Therefore we only need to look at the first two equations to solve the system:

$$\begin{cases} x_1 \quad - \quad x_3 = 13/3 \\ \quad x_2 - 2x_3 = 3. \end{cases}$$

Suppose we have chosen a value for x_1. Then the first equation gives us a value for x_3:

$$x_3 = x_1 - 13/3.$$

Plugging this into the second equation gives us

$$\begin{aligned} x_2 - 2(x_1 - 13/3) &= 3 \\ x_2 - 2x_1 + 26/3 &= 3 \\ x_2 &= 2x_1 - 17/3. \end{aligned}$$

So for any value x_1, we get the solution

$$(x_1, x_2, x_3) = (x_1,\ 2x_1 - 17/3,\ x_1 - 13/3)$$

The linear system therefore has infinite solutions, and so does the original system.

3.4 Echelon Forms

Now we have row-reduced linear systems that have one, zero and infinite solutions. What we actually did was to reduce the augmented matrix to *echelon form* matrices.

Definition 3.12 (Echelon Form)**.** A matrix is in echelon form if it satisfies the following three conditions:

1. All nonzero rows are above any rows of all zeros.
2. For each nonzero row, let its *leading entry* be the leftmost nonzery entry. Then each leading entry of a row is in a column to the right of the leading entry of the row above it. That is, every leading entry is strictly to the right of leading entries above it.
3. All entries directly below a leading entry are zero.

Definition 3.13 (Reduced echelon form)**.** A matrix is in reduced echelon form if it satisfies the three conditions above, and in addition satisfies the following:

4. Every leading entry is 1.
5. All entries directly above a leading entry are zero.

Example 3.14. Here are three examples of matrices in echelon form. If you want them in reduced echelon form, you need to replace every asterisk '*' directly above a leading entry, 1, by a 0.

$$\begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

Example 3.15. Here are three more matrices, two of which are not echelon.

$$\begin{bmatrix} 0 & 1 & 3 & -12 & 0 \\ 0 & 0 & 1 & 8 & 13 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Echelon

$$\begin{bmatrix} 0 & 1 & 3 & -12 & 0 \\ 0 & 0 & \boxed{0} & 1 & 1 \\ 0 & 0 & 1 & 8 & 13 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Not Echelon

$$\begin{bmatrix} 1 & 0 & 3 & -12 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 8 & 13 \\ 0 & \boxed{3} & 0 & 1 & 1 \end{bmatrix}$$

Not Echelon

Echelon and reduced echelon forms are useful to have, because every augmented matrix can be reduced to echelon form, and from there, we can easily determine the solution set of the original linear system. With a bit more terminology, we have a nice mathematical framework for describing solutions of linear systems:

3.5 Pivot Positions & Basic Variables

Consider a linear system with its augmented matrix.

Definition 3.16 (Pivot position)**.** A cell position in the matrix is a *pivot position* if it contains a leading entry in an echelon form of the matrix.

Definition 3.17 (Pivot column)**.** A *pivot column* is any matrix column that contains a pivot position.

Theorem 3.18.

1. *Every augmented matrix can be row reduced to an echelon form through elementary row operations.*

2. *The echelon form of a matrix is not unique, but its pivot positions are unique.*

3. *The reduced echelon form is unique.*

Recall that every column in the coefficient matrix corresponds to a variable: The first column contains the coefficients of x_1, the second column contains the coefficients of x_2, and so on...

Definition 3.19 (Basic variable)**.** For a linear system, we say that a variable x_i is a *basic variable* if its corresponding column is a pivot column.

Definition 3.20 (Free variable)**.** We say that a variable x_i is a *free variable* if it is not a basic variable.

3.6 Classifying Linear Systems

We now have all the terminology we need to make some statements about the solutions of linear systems in terms of their augmented matrices, pivot columns and echelon forms:

Proposition 3.21. *A linear system has no solutions if and only if the rightmost column of the augmented matrix is a pivot column.*

Another way of stating this proposition is that a linear system has no solutions if and only if the echelon form of the augmented matrix has a row with all zeros except for the rightmost entry.

Example 3.22. Recall that for the linear system

$$\begin{cases} \phantom{6x_1 - {}} 3x_2 - 6x_3 = 9 \\ 6x_1 - 9x_2 + 12x_3 = -1 \\ 5x_1 - 7x_2 + 9x_3 = 0 \end{cases}$$

we row reduced the augmented matrix until we got it in the form

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 5 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & -4 \end{array}\right].$$

This is in echelon form (but not reduced echelon form). We concluded that this system has no solutions, because the last row represents the equation

$$0 = -4$$

which is never true. But this is equivalent to saying that the rightmost column is a pivot column.

Theorem 3.23. *Suppose a linear system has at least one solution. Then it has exactly one solution if and only if all its variables are basic. Equivalently, it has infinite solutions if and only if one or more variables is free. Furthermore, the dimension of the solution set is equal to the number of free variables, or "degrees of freedom".*

To summarize: A linear system has

1. **No solutions, if and only if the rightmost column of its augmented matrix is a pivot column.**
2. **Otherwise, it has one unique solution if and only if all its variables are basic variables.**
3. **Likewise, it has infinitely many solutions if and only if it has a free variable.**
4. **The dimension of the solution set is equal to the number of free variables.**

Example 3.24. Consider the 4×4 linear system

$$\begin{cases} \quad\quad - 6x_2 - 3x_3 + 4x_4 = 9 \\ -3x_1 - \ \ x_2 - 5x_3 + 6x_4 = 0 \\ \ \ x_1 + 5x_2 + 4x_3 - 9x_4 = -7 \\ -2x_1 \quad\quad - 3x_3 + 3x_4 = -1. \end{cases}$$

This gives us the augmented matrix

$$\left[\begin{array}{cccc|c} 0 & -6 & -3 & 4 & 9 \\ -3 & -1 & -5 & 6 & 0 \\ 1 & 5 & 4 & -9 & -7 \\ -2 & 0 & -3 & 3 & -1 \end{array}\right].$$

We can row reduce this to get the echelon form

$$\left[\begin{array}{cccc|c} 1 & 5 & 4 & -9 & -7 \\ 0 & 2 & 1 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

So the pivot positions are $(1, 1)$, $(2, 2)$ and $(3, 4)$, which means the pivot columns are column 1, 2 and 4. Equivalently, the basic variables are x_1, x_2 and x_4, while the free variable is x_3. So we can conclude that this linear system has infinitely many solutions.

Once we have reduced the augmented matrix to echelon form, we can also find the solution set explicitly:

1. Firstly, the third row represents the equation

   $$x_4 = 0.$$

   So every solution will have $x_4 = 0$.

2. The first and second rows represent the linear equations

   $$\begin{cases} x_1 + 5x_2 + 4x_3 - 9x_4 = -7 \\ \quad\ \ 2x_2 + \ \ x_3 - 3x_4 = -3. \end{cases}$$

   Since $x_4 = 0$, this simplifies to

   $$\begin{cases} x_1 + 5x_2 + 4x_3 = -7 \\ \quad\ \ 2x_2 + \ \ x_3 = -3. \end{cases}$$

3. x_3 is a free variable, so we will let it be 'free'; i.e. we will only write it in terms of x_1 and x_2.

4. Suppose we have fixed a value for x_2. Then we can rearrange the second equation to get a value for x_3:

$$x_3 = -2x_2 - 3.$$

Plugging this into the first equation for x_3 gives us

$$\begin{aligned} x_1 + 5x_2 + 4\underbrace{x_3}_{=-2x_2-3} &= -7 \\ x_1 + 5x_2 + 4(-2x_2 - 3) &= -7 \\ x_1 + 5x_2 - 8x_2 - 12 &= -7 \\ x_1 &= 3x_2 + 5. \end{aligned}$$

5. So for each value of x_2, a solution $(x_1, x_2, x_3, 0)$ requires that

$$x_1 = 3x_2 + 5 \qquad \text{and} \qquad x_3 = -2x_2 - 3$$

6. Therefore, our solution set becomes

$$S = \big\{(3x_2 + 5,\ \ x_2,\ \ -2x_2 - 3,\ \ 0)\big\}$$

3.7 Algorithm for Row-reducing a Matrix

When row reducing a matrix to echelon form, some people are happy to just look at the matrix and apply row operations as they see fit, while others prefer to have an algorithm to follow. For the latter group, here is a short algorithm explaning how you can row reduce any matrix to reduced echelon form:

1. Start at the left-most column and the top row.
2. Denote the entry in this column and row by α.
   (a) If $\alpha \neq 0$:
      i. Scalar multiply this row by $1/\alpha$ to normalize α to 1.
      ii. By adding multiples of this row to other rows, set all other entries in this column to 0.
   (b) If $\alpha = 0$ but another entry in the column is non-zero, interchange the two rows so that the non-zero entry is above the zero entries. Then set this new non-zero entry to α and do step $2\,(a)$.
   (c) If $\alpha = 0$ and all other entries in the column are zero, then this column is not a pivot column, and you can't do anything with it.
3. Move one column to the right and repeat step 2. If the previous column was a pivot column, you move down one row to be below the pivot position. If the previous column was not a pivot column, you do not move down one row.
4. Repeat these steps until you have gone through all the columns.

If you still wonder how to row-reduce matrices and feel like this algorithm did not explain it well enough, there are plenty of websites that describe the algorithm in different ways. A quick google should give you something helpful!