(1)

Minors & Cofactors

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -7 & 5 \\ 1 & -1 & 6 \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} -7 & 5 \\ -1 & 6 \end{bmatrix}$$

$$M_{11} = \det(A_{11}) = -7 \cdot 6 - (-1) \cdot 5 = -42 + 5 = -37$$

$$C_{11} = (-1)^{1+1} M_{11} = 1 \cdot M_{11} = -37.$$

$$A_{32} = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$$

$$M_{32} = 3 \cdot 5 - 2 = 13$$

$$C_{32} = -M_{32} = -13$$

②

Cofactor

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 0 & 1 \\ 2 & 0 & -1 \end{bmatrix}.$$

Expand along column 2:

~~$\det(A) = a_{21}C_{21} + a_{22}C_{22} + a$~~

$$\det(A) = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$$

$$= 2 \cdot C_{12} + 0 + 0$$

$$= 2C_{12}.$$

$$C_{12} = (-1)^{1+2} \cdot \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix}$$

$$= -(3 \cdot (-1) - 2) = -(-5) = 5.$$

So $\det(A) = 2 \cdot 5 = 10.$

3

$-11+3(2)$

$$A=\begin{bmatrix}1 & 0 & 3\\ 2 & -5 & 1\\ 0 & 1 & 2\end{bmatrix},\quad \det(A)=-5$$

1. $R2 \to 4R2$

$$B=\begin{bmatrix}1 & 0 & 3\\ 8 & -20 & 4\\ 0 & 1 & 2\end{bmatrix},$$

$$\det(B)=4\cdot\det(A)=4\cdot(-5)=-20$$

2. $R3 \to R3+3R1$

$$C=\begin{bmatrix}1 & 0 & 3\\ 2 & -5 & 1\\ 3 & 1 & 11\end{bmatrix}$$

$$\det(C)=\det(A)=-5$$

3. $R1 \leftrightarrow R2$

$$D=\begin{bmatrix}2 & -5 & 1\\ 1 & 0 & 3\\ 0 & 1 & 2\end{bmatrix}$$

$$\det(D)=-\det(A)=-(-5)=5$$

(4)

Row Reduction

$$A = \begin{bmatrix} 2 & 8 & 4 & -6 \\ -2 & -5 & 2 & 11 \\ -1 & 2 & 3 & 13 \\ 1 & 4 & 2 & -5 \end{bmatrix}$$

Matrix	Row operation	Determinant
$\begin{bmatrix} 2 & 8 & 4 & -6 \\ -2 & -5 & 2 & 11 \\ -1 & 2 & 3 & 13 \\ 1 & 4 & 2 & -5 \end{bmatrix}$		$\lvert A\rvert$
$\begin{bmatrix} 1 & 4 & 2 & -3 \\ -2 & -5 & 2 & 11 \\ -1 & 2 & 3 & 13 \\ 1 & 4 & 2 & -5 \end{bmatrix}$	$R1 \to \frac{1}{2}R1$	$\frac{1}{2}\lvert A\rvert$
$\begin{bmatrix} 1 & 4 & 2 & -3 \\ 0 & 3 & 6 & 5 \\ 0 & 6 & 5 & 10 \\ 0 & 0 & 0 & -2 \end{bmatrix}$	$R2 \to R2+2R1$ $R3 \to R3+R1$ $R4 \to R4-R1$	$\frac{1}{2}\lvert A\rvert$
$\begin{bmatrix} 1 & 4 & 2 & -3 \\ 0 & 3 & 6 & 5 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$	$R3 \to R3-2R2$	$\frac{1}{2}\lvert A\rvert$

Upper-diagonal

So $\frac{1}{2}\lvert A\rvert = 1\cdot 3\cdot(-7)\cdot(-2) = 42$

So $\lvert A\rvert = 84.$

(5) Transpose:

$$A = \begin{bmatrix} 2 & -5 \\ 0 & -4 \\ 1 & 8 \end{bmatrix}, \quad A^T = \begin{bmatrix} 2 & 0 & 1 \\ -5 & -4 & 8 \end{bmatrix}$$

Finding adjugate ~~form~~ and inverse

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 1 \\ 5 & -1 & -1 \end{bmatrix}$$

1. Cofactor matrix:

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$= \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}$$

(6) $\left(A = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 2 & 1 \\ 5 & -1 & -1 \end{bmatrix}\right)$

$$\therefore C = \begin{bmatrix} \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 5 & -1 \end{vmatrix} \\ -\begin{vmatrix} 3 & -1 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 5 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 5 & -1 \end{vmatrix} \\ \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 6 & -11 \\ 4 & 4 & 16 \\ 5 & -2 & -1 \end{bmatrix}$$ Cofactor matrix.

3. $\operatorname{adj}(A) = C^T$

So

$$\operatorname{adj}(A) = \begin{bmatrix} -1 & 4 & 5 \\ 6 & 4 & -2 \\ -11 & 16 & -1 \end{bmatrix}$$

4. Find det: $\det(A)$.

Expand along 1st row

$$\begin{vmatrix} 1 & 3 & -1 \\ 1 & 2 & 1 \\ 5 & -1 & -1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 5 & -1 \end{vmatrix}$$

$$= -1 + 18 + 11 = 28 = \det(A)$$

(7) Use Cramer's rule

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & 2 \\ -1 & 2 & 1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 4 \\ 10 \\ -4 \end{bmatrix}$$

1. Find A_1, A_2, A_3 :

$$A_1 = \begin{bmatrix} 4 & 2 & 1 \\ 10 & 0 & 2 \\ -4 & 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 3 & 4 & 1 \\ 1 & 10 & 2 \\ -1 & -4 & 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 0 & 10 \\ -1 & 2 & -4 \end{bmatrix}.$$

2. Calculate $|A|, |A_1|, |A_2|, |A_3|$.

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 1 & 0 & 2 \\ -1 & 2 & 1 \end{vmatrix} = 3\begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix}$$

$$= 3(-4) - 2(3) + 2 = -16$$

8

Spanning

Consider

$$\underline{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \in \mathbb{R}^3$$

3 dimensions.

Linear combinations

Example: $3\underline{v}_1 + 2\underline{v}_2 = \begin{pmatrix}3\\0\\0\end{pmatrix} + \begin{pmatrix}0\\2\\0\end{pmatrix} = \underline{\begin{pmatrix}3\\2\\0\end{pmatrix}}$.

Spanning set: All possible linear combinations of $\underline{v}_1$ and $\underline{v}_2$.

$$S := \mathrm{Span}(\underline{v}_1, \underline{v}_2).$$

$$\begin{pmatrix}3\\2\\0\end{pmatrix} \in S, \quad \begin{pmatrix}4\\-1\\0\end{pmatrix} = 4\underline{v}_1 - \underline{v}_2 \in S$$

But $\begin{pmatrix}0\\0\\1\end{pmatrix} \notin S$.

So

$(\underline{v}_1, \underline{v}_2)$ do not span $\mathbb{R}^3$.

But $\left(\underline{v}_1, \underline{v}_2, \begin{pmatrix}0\\0\\1\end{pmatrix}\right)$ span $\mathbb{R}^3$.