1

3x3 Matrix Multiplication

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 4 & 0 & 2 \\ -3 & 2 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 3 & -1 \\ 5 & 0 & 7 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 3 & 2 & -1 \\ 4 & 0 & 2 \\ -3 & 2 & -2 \end{pmatrix} \begin{pmatrix} 0 & 1 & -2 \\ 1 & 3 & -1 \\ 5 & 0 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 3\cdot 0 + 2\cdot 1 - 1\cdot 5 & 3\cdot 1 + 2\cdot 3 - 1\cdot 0 & 3\cdot(-2) + 2\cdot(-1) - 1\cdot 7 \\ 4\cdot 0 + 0\cdot 1 + 2\cdot 5 & 4\cdot 1 + 0\cdot 3 + 2\cdot 0 & 4\cdot(-2) + 0\cdot(-1) + 2\cdot 7 \\ -3\cdot 0 + 2\cdot 1 - 2\cdot 5 & -3\cdot 1 + 2\cdot 3 - 2\cdot 0 & -3\cdot(-2) + 2\cdot(-1) - 2\cdot 7 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 9 & -15 \\ 10 & 4 & 6 \\ -8 & 3 & -10 \end{pmatrix}$$

Finding inverses through row reduction

1. $[A] \quad [I_n]$
2. Row reduce A, and for each row operation, do the same to I_n
3. If A is invertible, you can row reduce it to I_n.
4. Once $A \to I_n$, $I_n \to A^{-1}$.

$$\Downarrow$$

$$[A]\ [I_n]$$
$$\downarrow$$
$$[I_n]\ [A^{-1}]$$

② Invert $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$.

$$\left[\begin{matrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{matrix}\right] \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right]$$

$$\left[\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{matrix}\right] \left[\begin{matrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{matrix}\right] \quad \begin{matrix} R2 \to R2 - R1 \\ R3 \to R3 - R1 \end{matrix}$$

$$\left[\begin{matrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{matrix}\right] \left[\begin{matrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{matrix}\right] \quad \begin{matrix} R1 \to R1 - R2 \\ R3 \to R3 - 2R2 \end{matrix}$$

$$\left[\begin{matrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{matrix}\right] \left[\begin{matrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} \end{matrix}\right] \quad R3 \to \frac{1}{2} R3$$

$$\underbrace{\left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right]}_{\text{Since } I_n} \underbrace{\left[\begin{matrix} 3 & -3 & 1 \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{matrix}\right]}_{= A^{-1}} \quad \begin{matrix} R1 \to R1 + 2R3 \\ R2 \to R2 - 3R3 \end{matrix}$$

Matrix Equation:

Coefficient matrix:

$$A = \begin{bmatrix} 1 & -3 & 2 & -1 \\ -1 & 0 & 3 & 2 \\ 4 & 2 & 0 & -5 \end{bmatrix}$$

Variable vector

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

~~Con~~: Constant vector

$$\underline{b} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

$$A\underline{x} = \underline{b} : \begin{bmatrix} 1 & -3 & 2 & -1 \\ -1 & 0 & 3 & 2 \\ 4 & 2 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ +2 \end{bmatrix}$$

③ $A = \begin{bmatrix} 6 & 2 & 6 \\ 2 & 1 & 0 \\ -4 & -3 & 9 \end{bmatrix}, \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \underline{b} = \begin{bmatrix} 20 \\ 4 \\ 3 \end{bmatrix}.$

$$A\underline{x} = \underline{b}. \Rightarrow \underline{x} = A^{-1}\underline{b}.$$

Invert A:
$$\begin{bmatrix} 6 & 2 & 6 \\ 2 & 1 & 0 \\ -4 & -3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{3} & 1 \\ 2 & 1 & 0 \\ -4 & -3 & 9 \end{bmatrix} \begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R1 \to \tfrac{1}{6}R1$$

$$\begin{bmatrix} 1 & 1/3 & 1 \\ 0 & 1/3 & -2 \\ 0 & -5/3 & 13 \end{bmatrix} \begin{bmatrix} 1/6 & 0 & 0 \\ -1/3 & 1 & 0 \\ 2/3 & 0 & 1 \end{bmatrix} \quad \begin{aligned} R2 &\to R2 - 2R1 \\ R3 &\to R3 + 4R1 \end{aligned}$$

$$\begin{bmatrix} 1 & 1/3 & 1 \\ 0 & 1 & -6 \\ 0 & -5/3 & 13 \end{bmatrix} \begin{bmatrix} 1/6 & 0 & 0 \\ -1 & 3 & 0 \\ 2/3 & 0 & 1 \end{bmatrix} \quad R2 \to 3R2$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -1 & 0 \\ -1 & 3 & 0 \\ -1 & 5 & 1 \end{bmatrix} \quad \begin{aligned} R1 &\to R1 - \tfrac{1}{3}R2 \\ R3 &\to R3 + \tfrac{5}{3}R2 \end{aligned}$$

4 (cont...)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & \frac{1}{2} & -1 & 0 \\ 0 & 1 & -6 & -1 & 3 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{5}{3} & \frac{1}{3} \end{array}\right] \quad R3 \to \frac{1}{3}R3$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{= I_n} \underbrace{\begin{bmatrix} \frac{3}{2} & -6 & -1 \\ -3 & 13 & 2 \\ -\frac{1}{3} & \frac{5}{3} & \frac{1}{3} \end{bmatrix}}_{= A^{-1}} \quad \begin{array}{l} R1 \to R1 - 3R3 \\ R2 \to R2 + 6R3 \end{array}$$

$$\underline{x} = A^{-1}\underline{b}$$

$$\underline{x} = \begin{bmatrix} \frac{3}{2} & -6 & -1 \\ -3 & 13 & 2 \\ -\frac{1}{3} & \frac{5}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 20 \\ 4 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 30 - 24 - 3 \\ -60 + 52 + 6 \\ -\frac{20}{3} + \frac{20}{3} + 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Benefit of using this method:
Once you have A^{-1}, you can solve for any $\underline{b}$.

Showing Existence of Solutions

$$\cancel{\begin{aligned} 2x_1 + x_2 - x_3 &= b_1 \\ x_2 + x_3 &= b_2 \\ x_1 \quad + 2x_3 &= b_3 \end{aligned}}$$

Coefficient matrix: $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

column vectors: $\underline{a}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ $\underline{a}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\underline{a}_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Want to show

Standard unit vectors:

$$\left.\begin{matrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \checkmark \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{matrix}\right\} = c_1\underline{a}_1 + c_2\underline{a}_2 + c_3\underline{a}_3$$

1. $\underline{a}_1 - \underline{a}_3 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$.

   So $\frac{1}{2}\underline{a}_1 - \frac{1}{2}\underline{a}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

2.

$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$