1

$$\begin{cases} 2x_1 + x_2 = 3 \\ x_1 - x_2 = 2 \end{cases}$$ Solve by substitution

1. Rearrange 2nd equation to get x_2 alone

$x_1 - x_2 = 2$

$x_1 = x_2 + 2$

$\boxed{x_1 - 2 = x_2}$

2. Plug in for x_2 in 1st equation. Solve x_1.

$2x_1 + x_2 = 3$

$2x_1 + (x_1 - 2) = 3$

$3x_1 - 2 = 3$

$3x_1 = 5$

$\boxed{x_1 = \frac{5}{3}}$

3. Plug into equation x_1 to solve for x_2

$x_2 = x_1 - 2 = \left(\frac{5}{3}\right) - 2 = \frac{5}{3} - \frac{6}{3} = \frac{5-6}{3} = -\frac{1}{3}$.

4. So solution is $\boxed{x_1 = \frac{5}{3},\ x_2 = -\frac{1}{3}}$.

5. Check: $2x_1 + x_2 = 3 \Rightarrow 2\left(\frac{5}{3}\right) - \frac{1}{3} = \frac{10}{3} - \frac{1}{3} = \frac{9}{3} = 3$.

$x_1 - x_2 = 2 \Rightarrow \frac{5}{3} - \left(-\frac{1}{3}\right) = \frac{5}{3} + \frac{1}{3} = \frac{6}{3} = 2$.

Elementary row operations

1. Scaling: $c \neq 0$. $c = 2$.

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 1 \\ -1 & -3 & 4 & -1 \end{array}\right]$$

~~R1 → R2~~ $R1 \to 2R1$.

$$\to \left[\begin{array}{ccc|c} 6 & 2 & 4 & 2 \\ -1 & -3 & 4 & -1 \end{array}\right]$$

2. Adding rows:

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 1 \\ -1 & -3 & 4 & -1 \end{array}\right]$$

$R2 \to R2 + 3R1$

$$\to \left[\begin{array}{ccc|c} 3 & 1 & 2 & 1 \\ (-1+9) & (-3+3) & (4+6) & -1+3 \end{array}\right] = \left[\begin{array}{ccc|c} 3 & 1 & 2 & 1 \\ 8 & 0 & 10 & 2 \end{array}\right]$$

3. Interchange:

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 1 \\ -1 & -3 & 4 & -1 \end{array}\right]$$

$R1 \leftrightarrow R2$

$$\left[\begin{array}{ccc|c} -1 & -3 & 4 & -1 \\ 3 & 1 & 2 & 1 \end{array}\right]$$

② Solving system by Row Reduction

$$\begin{cases} x+y=3 \\ x-y=-1 \end{cases}.$$

1. Write augmented matrix:

$$\begin{cases} x+y=3 \\ x-y=-1 \end{cases} \qquad \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & -1 \end{array}\right]$$

2. Want zeros in matrix (coefficients):

$$\begin{cases} x+y=3 \\ \underbrace{2x+0y=2}_{2x=2} \end{cases} \qquad \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 0 & 2 \end{array}\right] \qquad R2 \to R2+R1$$

3. $$\begin{matrix} x+y=3 \\ x \quad =1 \end{matrix} \qquad \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & 0 & 1 \end{array}\right] \qquad R2 \to \frac{1}{2}R2$$

4. $$\begin{cases} x \quad =1 \\ x+y=3 \end{cases} \qquad \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 1 & 3 \end{array}\right] \qquad R1 \leftrightarrow R2$$

5. $$\begin{cases} x=1 \\ y=2 \end{cases} \qquad \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \qquad R2 \to R2-R1$$

$$\boxed{\text{SO} \quad x=1,\ y=2}$$

comments:

- We aim to reduce coefficient matrix to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solving using row reduction

$$\begin{cases} x_1 + 2x_2 - x_3 = 2 \\ 2x_1 - x_2 + 5x_3 = 15 \\ x_1 \quad\quad + 3x_3 = 10 \end{cases} ;$$

Want zeros

1.
$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & -1 & 5 & 15 \\ 1 & 0 & 3 & 10 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ (2-2\cdot 1) & (-1-2\cdot 2) & (5-(2)(-1)) & 15-2\cdot 2 \\ (1-1) & (0-2) & (3-(-1)) & 10-2 \end{array}\right] \begin{array}{l} R2 \to R2 - 2R1 \\ R3 \to R3 - R1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -5 & 7 & 11 \\ 0 & -2 & 4 & 8 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -2 & 4 & 8 \\ 0 & -5 & 7 & 11 \end{array}\right] \quad R2 \leftrightarrow R3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -2 & -4 \\ 0 & -5 & 7 & 11 \end{array}\right] \quad R2 \to -\frac{1}{2}R2$$

$$\left[\begin{array}{ccc|c} 1 & 2-2 & -1-(-2)2 & 2-(-2)4 \\ 0 & 1 & -2 & -4 \\ 0 & -5+5 & 7+5(-2) & 11+5(-4) \end{array}\right] \begin{array}{l} R1 \to R1 - 2R2 \\ R3 \to R3 + 5R2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & -3 & -9 \end{array}\right]$$

(3) (cont...)

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & -3 & -9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad R3 \to -\frac{1}{3}R3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \begin{array}{l} R1 \to R1 - 3R3 \\ R2 \to R2 + 2R3 \end{array}$$

$$x_1 = 1$$
$$x_2 = 2$$
$$x_3 = 3$$

No solutions

Row red. $\left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 9 \\ 5 & -7 & 9 & 0 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 9 \\ 0 & 3 & -6 & 5 \end{array}\right]$ $R3 \to R3 - 5R1$

$$\boxed{\begin{array}{c} 3x_2 - 6x_3 = 9 \\ 3x_2 - 6x_3 = 5. \end{array}}$$

$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 9 \\ 0 & 0 & 0 & -4 \end{array}\right]$ $R3 \to R3 - R2$

R3 : $0x_1 + 0x_2 + 0x_3 = -4$

$0 = -4.$

So no solutions.

Impossible to get $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

4

Infinite solutions

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 9 \\ 5 & -7 & 9 & 4 \end{array}\right]$$

Used to be 0

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 9 \\ 0 & 3 & -6 & 9 \end{array}\right] R3 \to R3 - 5R1$$

R2 & R3: $3x_2 - 6x_3 = 9$.

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right] R3 \to R3 - R2$$

R3: $0 = 0$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] R2 \to \tfrac{1}{3} R2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 5 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] R1 \to R1 + 2R2$$

Solution set:

$x_1 - x_3 = 5$

$x_2 - 2x_3 = 3$

Fix x_1.

$$x_1 - x_3 = 5$$

$$\boxed{x_3 = x_1 - 5.}$$

$$x_2 - 2x_3 = 3$$

$$x_2 = 3 + 2x_3 = 3 + 2(x_1 - 5)$$

$$= 3 + 2x_1 - 10$$

$$\boxed{x_2 = 2x_1 - 7}$$

Solution: $\{(x_1, 2x_1 - 7, x_1 - 5) = (x_1, x_2, x_3)\}$

⑤ Pivot position:

Pivot positions

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & -1 & 5 & 15 \\ 1 & 0 & 3 & 10 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

~~Pivot~~ Pivot positions: (1,1), (2,2), (3,3)
(Row, Column)

Pivot columns: Column 1, 2 & 3.

Basic variables: x_1, x_2, x_3

Free variables: $\emptyset$

Pivot column on the right:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 9 \\ 5 & -7 & 9 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 9 \\ 0 & 0 & 0 & -4 \end{array}\right]$$

R3: $0 = -4$.

⑥

Algorithm for row reduction

1. Move up rows s.t. you have non-zero value furthest up, and furthest left.

$$\left[\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 3 & 4 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 1 & 2 \end{array}\right]$$

2. Cancel out all entries directly below entry up to left.

$$\left[\begin{array}{cc|c} 1 & 3 & 4 \\ \cancel{3}2 & 1 & 2 \end{array}\right]$$

3. Once done, ignore leftmost column ~~row~~, and do the same to the neighbouring column on the right.*

Repeat step 1 & 2

* and ignore top row and go one row down

1. $$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 2 \\ 3 & 3 & 1 & 4 \\ 1 & 6 & 1 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 6 & 1 & -2 \\ 3 & 3 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

2. $$0 \leftarrow \left[\begin{array}{ccc|c} 1 & 6 & 1 & -2 \\ 3 & 3 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R2-3R1} \left[\begin{array}{ccc|c} 1 & 6 & 1 & -2 \\ 0 & -15 & -2 & 10 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

3. $$\left[\begin{array}{ccc|c} 1 & 6 & 1 & -2 \\ 0 & -15 & -2 & 10 \\ 0 & 0 & 1 & 2 \end{array}\right]$$