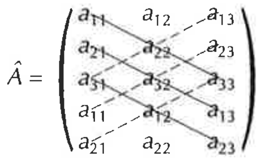

Computing the determinant of a 3×3 *matrix.* **Figure 9.1**

> **Theorem 9.3** A square matrix is nonsingular if and only if its determinant is nonzero.

Proof Sketch Recall that a square matrix A is nonsingular if and only if its row echelon form R has no all-zero rows. Since each row of the square matrix R has more leading zeros than the previous row, R has no all-zero rows if and only if the jth row of R has exactly $(j-1)$ leading zeros. This occurs if and only if R has no zeros on its diagonal. Since $\det R$ is the product of its diagonal entries, A is nonsingular if and only if $\det R$ is nonzero. Since $\det R = \pm \det A$, A is nonsingular if and only if $\det A$ is nonzero. ■

Theorem 9.3 is obvious for 1×1 matrices, because the equation $ax = b$ has a unique solution, $x = b/a$, for every b if and only if $a \neq 0$. Theorem 8.8 demonstrates Theorem 9.3 for 2×2 matrices.

EXERCISES

9.1 Write out the complete expression for the determinant of a 3×3 matrix — six terms, each a product of three entries.

9.2 Write out the definition of the determinant of a 4×4 matrix in terms of the determinants of certain of its 3×3 submatrices. How many terms are there in the complete expansion of the determinant of a 4×4 matrix?

9.3 Compute out the expression on the right-hand side of (5). Show that it equals the expression calculated in Exercise 9.1.

9.4 Show that one obtains the same formula for the determinant of a 2×2 matrix, no matter which row or column one uses for the expansion.

9.5 Use a formula for the determinant to verify Theorem 9.1 for upper-triangular 3×3 matrices.

9.6 Verify the conclusion of Theorem 9.2 for 2×2 matrices by showing that the determinant of a general 2×2 matrix is not changed if one adds r times row 1 to row 2.

9.7 For each of the following matrices, compute the row echelon form and verify the conclusion of Theorem 9.2:

$$a)\ \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \qquad b)\ \begin{pmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{pmatrix}, \qquad c)\ \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 0 & 7 & 8 \end{pmatrix}.$$

9.8 Use the observation following Theorem 9.2 to carry out a quick calculation of the determinant of each of the following matrices:

$$a)\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 4 & 3 \end{pmatrix}, \qquad b)\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 5 \\ 1 & 9 & 6 \end{pmatrix}.$$

9.9 Use Theorem 9.3 to determine which of the matrices in Exercises 9.7 and 9.8 are nonsingular.

9.2 USES OF THE DETERMINANT

Since the determinant tells whether or not A^{-1} exists and whether or not $A\mathbf{x} = \mathbf{b}$ has a unique solution, it is not surprising that one can use the determinant to derive a formula for A^{-1} and a formula for the solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$. First, we define the adjoint matrix of A as the transpose of the matrix of cofactors of A.

Definition For any $n \times n$ matrix A, let C_{ij} denote the (i, j)th cofactor of A, that is, $(-1)^{i+j}$ times the determinant of the submatrix obtained by deleting row i and column j from A. The $n \times n$ matrix whose (i, j)th entry is C_{ji}, the (j, i)th cofactor of A (note the switch in indices), is called the **adjoint** of A and is written **adj** A.

Theorem 9.4 Let A be a nonsingular matrix. Then,

(*a*) $A^{-1} = \dfrac{1}{\det A} \cdot \operatorname{adj} A$, and

(*b*) (**Cramer's rule**) the unique solution $\mathbf{x} = (x_1, \cdots, x_n)$ of the $n \times n$ system $A\mathbf{x} = \mathbf{b}$ is

$$x_i = \frac{\det B_i}{\det A}, \qquad \text{for } i = 1, \ldots, n,$$

where B_i is the matrix A with the right-hand side $\mathbf{b}$ replacing the ith column of A.

For 3×3 systems,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3. \end{aligned}$$

Finally, we note three algebraic properties of the determinant function which we will find important in our use of determinants.

> **Theorem 9.5** Let A be a square matrix. Then,
>
> (*a*) $\det A^T = \det A$,
> (*b*) $\det(A \cdot B) = (\det A)(\det B)$, and
> (*c*) $\det(A + B) \neq \det A + \det B$, in general.

Gaussian elimination is a much more efficient method of solving a system of n equations in n unknowns than is Cramer's rule. Cramer's rule requires the evaluation of $(n+1)$ determinants. Each determinant is a sum of $n!$ terms and each term is a product of n entries. So, Cramer's rule requires $(n+1)!$ operations. On the other hand, the number of arithmetic operations required by Gaussian elimination for such a system is on the order of n^3. If $n = 6$ as in the Leontief model in Section 8.5, then $(n+1)!$ is 5040, while n^3 is only 216; the difference grows exponentially as n increases.

Nevertheless, Cramer's rule is particularly useful for small linear systems in which the coefficients a_{ij} are parameters and for which one wants to obtain a general formula for the endogenous variables (the x_i's) in terms of the parameters and the exogenous variables (the b_j's). One can then see more clearly how changes in the parameters affect the values of the endogenous variables.

EXERCISES

9.10 Verify directly that matrix (9) really is the inverse of matrix (8) in Example 9.3.

9.11 Use Theorem 9.4 to invert the following matrices:

$$a)\ \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}, \qquad b)\ \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{pmatrix}, \qquad c)\ \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

9.12 Use Cramer's rule to compute x_1 and x_2 in Example 9.4.

9.13 Use Cramer's rule to solve the following systems of equations:

$$a)\ \begin{aligned} 5x_1 + x_2 &= 3 \\ 2x_1 - x_2 &= 4; \end{aligned} \qquad b)\ \begin{aligned} 2x_1 - 3x_2 \qquad &= 2 \\ 4x_1 - 6x_2 + x_3 &= 7 \\ x_1 + 10x_2 \qquad &= 1. \end{aligned}$$

9.14 Verify the conclusions of Theorem 9.5 for the following pairs of matrices:

$$a)\ A = \begin{pmatrix} 4 & 5 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix};$$

b) $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix};$

c) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$

9.3 IS-LM ANALYSIS VIA CRAMER'S RULE

As an illustrative example, consider the linear IS-LM national income model described in Chapter 6:

$$\begin{aligned} sY + ar &= I^o + G \\ mY - hr &= M_s - M^o \end{aligned} \tag{10}$$

where Y = net national product
r = interest rate
s = marginal propensity to save,
a = marginal efficiency of capital,
I = investment $(= I^o - ar)$,
m = money balances needed per dollar of transactions,
G = government spending,
M_s = money supply.

All the parameters are positive. Because the coefficients in this system are parameters instead of numbers, it is easiest to solve (10) using Cramer's rule:

$$Y = \frac{\begin{vmatrix} I^o + G & a \\ M_s - M^o & -h \end{vmatrix}}{\begin{vmatrix} s & a \\ m & -h \end{vmatrix}} = \frac{(I^o + G)h + a(M_s - M^o)}{sh + am}$$

$$r = \frac{\begin{vmatrix} s & I^o + G \\ m & M_s - M^o \end{vmatrix}}{\begin{vmatrix} s & a \\ m & -h \end{vmatrix}} = \frac{(I^o + G)m - s(M_s - M^o)}{sh + am}.$$

One can now use these expressions to see that, in this model, an increase in I^o, G, or M_s or a decrease in M^o or m will lead to an increase in the equilibrium net product Y. An increase in I^o or M^o or a decrease in M_s, h, or m will lead to an increase in equilibrium interest rate r.

8.43 *a)* $\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 0 & -1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 6 \\ 0 & 0 & 3 \end{pmatrix}$,

c) $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 & 0 & 1 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 3 & 8 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 6 & 0 & 5 \\ 0 & 3 & 8 & 2 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

8.48 *a)* $\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$,

b) $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{pmatrix}$,

8.51 *b)* $(1,0,-1)$, $(-1,1,2)$, $(1,1,-1)$, $(0,1,-1)$.

Chapter 9 Answers

9.1 $a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}$.

9.2 $a_{11} \cdot \det\begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix} - a_{12}\det\begin{pmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{pmatrix}$

$+ a_{13}\det\begin{pmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{pmatrix} - a_{14}\det\begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$.

9.5 $\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = a_{11} \cdot \det\begin{pmatrix} a_{22} & a_{23} \\ 0 & a_{33} \end{pmatrix} - 0 \cdot \det\begin{pmatrix} a_{12} & a_{13} \\ 0 & a_{33} \end{pmatrix}$

$+ 0 \cdot \det\begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix} = a_{11}a_{22}a_{33} + 0 + 0$, expanding along column one.

9.6 $\det\begin{pmatrix} a_{11} & a_{12} \\ ra_{11} + a_{21} & ra_{12} + a_{22} \end{pmatrix} = a_{11}(ra_{12} + a_{22}) - a_{12}(ra_{11} + a_{21})$

$= ra_{11}a_{12} - ra_{11}a_{12} + a_{11}a_{22} - a_{12}a_{21}$.

9.7 *a)* $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, determinants $= -1$.

b) $\begin{pmatrix} 2 & 4 & 0 \\ 0 & -8 & 3 \\ 0 & 0 & 3/4 \end{pmatrix}$, determinants $= -12$.

c) One row echelon form is $\begin{pmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & -6 \end{pmatrix}$, with det $= -18 = -\det A$.

9.8 *a*) One row echelon form is $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. So, det $= 3$.

b) One row echelon form is $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 5 \\ 0 & 0 & -5 \end{pmatrix}$. So, det $= -20$.

9.9 All nonsingular since det $\neq 0$.

9.11 *a*) $\frac{1}{1}\begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}$.

b) $$\frac{1}{\det A} \cdot \begin{pmatrix} \begin{vmatrix} 5 & 6 \\ 0 & 8 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 0 & 8 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \\ -\begin{vmatrix} 0 & 6 \\ 1 & 8 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 8 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & 6 \end{vmatrix} \\ \begin{vmatrix} 0 & 5 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} \end{pmatrix}$$

$$= \frac{1}{37} \cdot \begin{pmatrix} 40 & -16 & -3 \\ 6 & 5 & -6 \\ -5 & 2 & 5 \end{pmatrix}.$$

c) $\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

9.12 $x_1 = \frac{35}{35} = 1, \quad x_2 = -\frac{70}{35} = -2$

9.13 *a*) $x_1 = \frac{-7}{-7} = 1,\ x_2 = \frac{14}{-7} = -2.$

b) $x_1 = \frac{-23}{-23} = 1;\ x_2 = \frac{0}{-23} = 0;\ x_3 = \frac{-69}{-23} = 3.$

9.14 *a*) $\det A = -1$, $\det B = -1$, $\det AB = +1$; $\det(A + B) = -4$.
b) $\det A = 24$; $\det B = 18$; $\det AB = 432$; $\det(A + B) = 56$.
c) $\det A = ad - bc$, $\det B = eh - fg$, $\det AB = (ad - bc)(eh - fg)$,
$\det(A + B) = \det A + \det B + ah - bg + de - cf$.

Chapter 10 Answers

10.4 *a*) $(2, -1)$ *b*) $(-2, -1)$ *c*) $(2, 1)$ *d*) $(3, 0)$ *e*) $(1, 2, 4)$ *f*) $(2, -2, 3)$.

10.5 *a*) $(1, 3)$ *b*) $(-4, 12)$ *c*) undefined *d*) $(0, 3, 3)$ *e*) $(0, 2)$
f) $(1, 4)$ *g*) $(1, 1)$ *h*) $(3, 7, 1)$ *i*) $(-2, -4, 0)$ *j*) undefined

10.10 *a*) 5 *b*) 3 *c*) $\sqrt{3}$ *d*) $3\sqrt{2}$ *e*) $\sqrt{2}$ *f*) $\sqrt{14}$ *g*) 2 *h*) $\sqrt{30}$ *i*) 3

10.11 *a*) 5 *b*) 10 *c*) 4 *d*) $\sqrt{41}$ *e*) 6