nonzero coefficient is 1:

$$
\begin{aligned}
x_1 - 0.4x_2 - 0.3\ x_3 &= 130 \\
x_2 - 0.25x_3 &= 125 \\
x_3 &= 300.
\end{aligned} \tag{11}
$$

Now, instead of using back substitution, use Gaussian elimination methods from the *bottom* equation to the top to eliminate all but the first term on the left-hand side in each equation in (11). For example, add 0.25 times equation (11c) to equation (11b) to eliminate the coefficient of $x_3$ in (11b) and obtain $x_2 = 200$. Then, add 0.3 times (11c) to (11a) and 0.4 times (11b) to (11a) to obtain the new system:

$$
\begin{aligned}
x_1 \qquad\qquad &= 300 \\
x_2 \qquad &= 200 \\
x_3 &= 300,
\end{aligned} \tag{12}
$$

which needs no further work to see the solution. Gauss-Jordan elimination is particularly useful in developing the theory of linear systems; Gaussian elimination is usually more efficient in solving actual linear systems.

Earlier we mentioned a third method for solving linear systems, namely matrix methods. We will study these methods in the next two chapters, when we discuss matrix inversion and Cramer's rule. For now, it suffices to note that all the intuition behind these more advanced methods derives from Gaussian elimination. The understanding of this technique will provide a solid base on which to build your knowledge of linear algebra.

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## EXERCISES

**7.1** Which of the following equations are linear?

*a)* $3x_1 - 4x_2 + 5x_3 = 6$; *b)* $x_1x_2x_3 = -2$; *c)* $x^2 + 6y = 1$;
*d)* $(x + y)(x - z) = -7$; *e)* $x + 3^{1/2}z = 4$; *f)* $x + 3z^{1/2} = -4$.

**7.2** Solve the following systems by substitution, Gaussian elimination, and Gauss-Jordan elimination:

*a)*
$$
\begin{aligned}
x - 3y + 6z &= -1 \\
2x - 5y + 10z &= 0 \\
3x - 8y + 17z &= 1;
\end{aligned}
$$

*b)*
$$
\begin{aligned}
x_1 + x_2 + x_3 &= 0 \\
12x_1 + 2x_2 - 3x_3 &= 5 \\
3x_1 + 4x_2 + x_3 &= -4.
\end{aligned}
$$

**7.3** Solve the following systems by Gauss-Jordan elimination. Note that the third system requires an equation interchange.

*a*) $\begin{aligned} 3x + 3y &= 4 \\ x - y &= 10; \end{aligned}$ *b*) $\begin{aligned} 4x + 2y - 3z &= 1 \\ 6x + 3y - 5z &= 0 \\ x + y + 2z &= 9; \end{aligned}$ *c*) $\begin{aligned} 2x + 2y - z &= 2 \\ x + y + z &= -2 \\ 2x - 4y + 3z &= 0. \end{aligned}$

**7.4** Formalize the three elementary equation operations using the abstract notation of system (2), and for each operation, write out the operation which reverses its effect.

**7.5** Solve the IS-LM system in Exercise 6.7 by substitution.

**7.6** Consider the general IS-LM model with no fiscal policy in Chapter 6. Suppose that $M_s = M^o$; that is, the intercept of the LM-curve is 0.

*a*) Use substitution to solve this system for $Y$ and $r$ in terms of the other parameters.

*b*) How does the equilibrium GNP depend on the marginal propensity to save?

*c*) How does the equilibrium interest rate depend on the marginal propensity to save?

**7.7** Use Gaussian elimination to solve

$$\begin{cases} 3x + 3y = 4 \\ -x - y = 10. \end{cases}$$

What happens and why?

**7.8** Solve the general system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2. \end{cases}$$

What assumptions do you have to make about the coefficients $a_{ij}$ in order to find a solution?

## 7.2 ELEMENTARY ROW OPERATIONS

The focus of our concern in the last section was on the coefficients $a_{ij}$ and $b_i$ of the systems with which we worked. In fact, it was a little inefficient to rewrite the $x_i$'s, the plus signs, and the equal signs each time we transformed a system. It makes sense to simplify the representation of linear system (2) by writing two rectangular arrays of its coefficients, called **matrices**. The first array is

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

For example, multiply the second row of (14) by $1/0.8$ and the third row of (14) by $1/0.7$ to achieve the matrix

$$\begin{pmatrix} 1 & -0.4 & -0.3 & | & 130 \\ 0 & 1 & -0.25 & | & 125 \\ 0 & 0 & 1 & | & 300 \end{pmatrix}.$$

Then, use the pivot in row 3 to turn the entries $-0.25$ and $-0.3$ above it into zeros — first by adding 0.25 times row 3 to row 2 and then by adding 0.3 times row 3 to row 1. The result is

$$\begin{pmatrix} 1 & -0.4 & 0 & | & 220 \\ 0 & 1 & 0 & | & 200 \\ 0 & 0 & 1 & | & 300 \end{pmatrix}.$$

Finally, use the pivot in row 2 to eliminate the nonzero entry above it by adding 0.4 times row 2 to row 1 to get the matrix

$$\begin{pmatrix} 1 & 0 & 0 & | & 300 \\ 0 & 1 & 0 & | & 200 \\ 0 & 0 & 1 & | & 300 \end{pmatrix}. \qquad (15)$$

Notice that this is the augmented matrix for system (12) and that one can read the solution right off the last column of this matrix:

$$x_1 = 300, \qquad x_2 = 200, \qquad x_3 = 300.$$

We say that matrix (15) is in *reduced row echelon form.*

**Definition** A row echelon matrix in which each pivot is a 1 and in which each column containing a pivot contains no other nonzero entries is said to be in **reduced row echelon form.**

The matrices in Examples 7.4 and 7.5 above are in reduced row echelon form. Note that in transforming a matrix to row echelon form we work from top left to bottom right. To achieve the reduced row echelon form, we continue in the same way but in the other direction, from bottom right to top left.

## EXERCISES

**7.9** Describe the row operations involved in going from equations (8) to (10).

**7.10** Put the matrices in Examples 7.2 and 7.3 in reduced row echelon form.

**7.11** Write the three systems in Exercise 7.3 in matrix form. Then use row operations to find their corresponding row echelon and reduced row echelon forms and to find the solution.

**7.12** Use Gauss-Jordan elimination in matrix form to solve the system

$$\begin{aligned} w + x + 3y - 2z &= 0 \\ 2w + 3x + 7y - 2z &= 9 \\ 3w + 5x + 13y - 9z &= 1 \\ -2w + x \qquad - z &= 0. \end{aligned}$$

## 7.3 SYSTEMS WITH MANY OR NO SOLUTIONS

As we will study in more detail later, the locus of all points $(x_1, x_2)$ which satisfy the linear equation $a_{11}x_1 + a_{12}x_2 = b_1$ is a straight line in the plane. Therefore, the solution $(x_1, x_2)$ of the two linear equations in two unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned} \tag{16}$$

is a point which lies on both lines of (16) in the Cartesian plane. Solving system (16) is equivalent to finding where the two lines given by (16) cross. In general, two lines in the plane will be nonparallel and will cross in exactly one point. However, the lines given by (16) can be parallel to each other. In this case, they will either coincide or they will never cross. If they coincide, every point on either line is a solution to (16); and (16) has *infinitely* many solutions. An example is the system

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 2x_1 + 4x_2 &= 6. \end{aligned}$$

In the case where the two parallel lines do not cross, the corresponding system has *no* solution, as the example

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ x_1 + 2x_2 &= 4 \end{aligned}$$

illustrates. Therefore, it follows from geometric considerations that two linear equations in two unknowns can have one solution, no solution, or infinitely many solutions. We will see later in this chapter that this principle holds for every system of *m linear* equations in *n* unknowns.

$$\begin{pmatrix} * & w & w & w & w & w & w & | & w \\ 0 & 0 & 0 & * & w & w & w & | & w \\ 0 & 0 & 0 & 0 & * & w & w & | & w \\ 0 & 0 & 0 & 0 & 0 & 0 & * & | & w \end{pmatrix}.$$

This matrix is in row echelon form. The corresponding reduced row echelon form is

$$\begin{pmatrix} 1 & w & w & 0 & 0 & w & 0 & | & w \\ 0 & 0 & 0 & 1 & 0 & w & 0 & | & w \\ 0 & 0 & 0 & 0 & 1 & w & 0 & | & w \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & | & w \end{pmatrix}.$$

The final solution will have the form

$$\begin{aligned} x_1 &= a_1 - a_2x_2 - a_3x_3 - a_4x_6, \\ x_4 &= b_1 \qquad\qquad\qquad\quad - b_2x_6, \\ x_5 &= c_1 \qquad\qquad\qquad\quad - c_2x_6, \\ x_7 &= d_1. \end{aligned}$$

Here $x_7$ is the only variable which is unambiguously determined. The variables $x_2$, $x_3$, and $x_6$ are free to take on any values; once values have been selected for these three variables, then values for $x_1$, $x_4$, and $x_5$ are automatically determined.

Some more vocabulary is helpful here. If the $j$th column of the row echelon matrix $\hat{B}$ contains a pivot, we call $x_j$ a **basic variable**. If the $j$th column of $\hat{B}$ does not contain a pivot, we call $x_j$ a **free** or **nonbasic variable**. In this terminology, Gauss-Jordan elimination determines a solution of the system in which each basic variable is either unambiguously determined or a linear expression of the free variables. The free variables are free to take on any value. Once one chooses values for the free variables, values for the basic variables are determined.

As in the example above, the free variables are often placed on the right-hand side of the equations to emphasize that their values are not determined by the system; rather, they act as parameters in determining values for the basic variables.

In a given problem which variables are free and which are basic may depend on the order of the operations used in the Gaussian elimination process and on the order in which the variables are indexed.

## EXERCISES

**7.13** Reduce the following matrices to row echelon and reduced row echelon forms:

$$a)\ \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}, \qquad b)\ \begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 & 7 \end{pmatrix}, \qquad c)\ \begin{pmatrix} -1 & -1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

**7.14** Solve the system of equations $\begin{cases} -4x + 6y + 4z = 4 \\ 2x - y + z = 1. \end{cases}$

**7.15** Use Gauss-Jordan elimination to determine for what values of the parameter $k$ the system

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - kx_2 &= 1 \end{aligned}$$

has no solutions, one solution, and more than one solution.

**7.16** Use Gauss-Jordan elimination to solve the following four systems of linear equations. Which variables are free and which are basic in each solution?

$$a)\ \begin{aligned} w + 2x + y - z &= 1 \\ 3w - x - y + 2z &= 3 \\ -x + y - z &= 1 \\ 2w + 3x + 3y - 3z &= 3; \end{aligned} \qquad b)\ \begin{aligned} w - x + 3y - z &= 0 \\ w + 4x - y + z &= 3 \\ 3w + 7x + y + z &= 6 \\ 3w + 2x + 5y - z &= 3; \end{aligned}$$

$$c)\ \begin{aligned} w + 2x + 3y - z &= 1 \\ -w + x + 2y + 3z &= 2 \\ 3w - x + y + 2z &= 2 \\ 2w + 3x - y + z &= 1; \end{aligned} \qquad d)\ \begin{aligned} w + x - y + 2z &= 3 \\ 2w + 2x - 2y + 4z &= 6 \\ -3w - 3x + 3y - 6z &= -9 \\ -2w - 2x + 2y - 4z &= -6. \end{aligned}$$

**7.17** *a*) Use the flexibility of the free variable to find *positive integers* which satisfy the system

$$\begin{aligned} x + y + z &= 13 \\ x + 5y + 10z &= 61. \end{aligned}$$

*b*) Suppose you hand a cashier a dollar bill for a 6-cent piece of candy and receive 16 coins as your change — all pennies, nickels, and dimes. How many coins of each type do you receive? [Hint: See part *a*.]

**7.18** For what values of the parameter $a$ does the following system of equations have a solution?

$$\begin{aligned} 6x + y &= 7 \\ 3x + y &= 4 \\ -6x - 2y &= a. \end{aligned}$$

**7.19** From Chapter 6, the stationary distribution in the Markov model of unemployment satisfies the linear system

$$\begin{aligned} (q - 1)x + py &= 0 \\ (1 - q)x - py &= 0 \\ x + y &= 1. \end{aligned}$$

*a*) If $p$ and $q$ lie between 0 and 1, how many solutions does this system have? Why?
*b*) Ignoring the condition that $p$ and $q$ must be between and 0 and 1, find values of $p$ and $q$ so that this system has no solutions.

## 7.4 RANK — THE FUNDAMENTAL CRITERION

We now answer the five basic questions about existence and uniqueness of solutions that were posed in Section 7.3. The main criterion involved in the answers to these questions is the rank of a matrix. First, note that we say a row of a matrix is nonzero if and only if it contains at least one nonzero entry.

**Definition** The **rank** of a matrix is the number of nonzero rows in its row echelon form.

Since we can reduce any matrix to several different row echelon matrices (if we interchange rows), we need to show that this definition of rank is independent of which row echelon matrix we compute. We will save this for Chapter 27, where we will also discuss the rank of a matrix from a different, more geometric point of view.

Let $A$ and $\hat{A}$ be the coefficient matrix and augmented matrix respectively of a system of linear equations. Let $B$ and $\hat{B}$ be their corresponding row echelon forms. One goes through the same steps in reducing $A$ to $B$ as in reducing $\hat{A}$ to $\hat{B}$ no matter what the last column of $\hat{A}$ is, because the choices of elementary row operations in going from $\hat{A}$ to $\hat{B}$ never involve the last column of the augmented matrix. In other words, $\hat{B}$ is itself an augmented matrix for $B$.

We first relate the rank of a coefficient matrix $A$ to the rank of a corresponding augmented matrix and to the number of rows and columns of $A$. Note that the rank of the augmented matrix must be at least as big as the rank of the coefficient matrix because if a row in the augmented matrix contains only zeros, then so does the corresponding row of the coefficient matrix. Furthermore, the definition of rank requires that the rank is less than or equal to the number of rows of the coefficient matrix. Since each nonzero row in the row echelon form contains exactly one pivot, the rank is equal to the number of pivots. Since each column of $A$ can have at most one pivot, the rank is also less than or equal to the number of columns of the coefficient matrix. Fact 7.1 summarizes the observations in this paragraph.

**Fact 7.1.** Let $A$ be the coefficient matrix and let $\hat{A}$ be the corresponding augmented matrix. Then,

(*a*) $\text{rank} A \leq \text{rank} \hat{A}$,
(*b*) $\text{rank} A \leq$ number of rows of $A$, and
(*c*) $\text{rank} A \leq$ number of columns of $A$.

The following fact relates the ranks of $A$ and of $\hat{A}$ to the existence of a solution of the system in question and gives us our first answer to Question 1 above.

*e*) $f'(x) = \frac{1}{\ln x} - \frac{1}{(\ln x)^2}$ $f''(x) = \frac{2}{x(\ln x)^3} - \frac{1}{x(\ln x)^2}$
*f*) $f'(x) = (1 - \ln x)/x^2$ $f''(x) = (2\ln x - 3)/x^3$

**5.10** $dy/dx = 1/(x \ln 10)$

**5.11** $c$

**5.12** *a*) \$1870.62 *b*) \$4754.17

**5.13** 48.05 years

**5.14** $t = 1/(4r^2)$

**5.15** $t = 3.393$

**5.16** *a*) $y' = 3x/(x^2 + 1)^{1/2}(x^2 + 4)^{3/2}$
*b*) $y' = 2x(\ln(x^2) + 1)(x^2)^{x^2}$

## Chapter 6 Answers

**6.1** Net contribution cost = \$5956.

**6.2** $C = 6070$, $S = 2875$, $F = 36{,}422$.

**6.3** $x_1 = 0.5x_1 + 0.5x_2 + 1$, $x_2 = 0x_1 + 0.25x_2 + 3$; $x_1 = 6, x_2 = 4$.

**6.4** $x_1 = 0.5x_1 + 0.5x_2 + 1$, $x_2 = 0.875x_1 + 0.25x_2 + 3$; no positive solution.

**6.5** 8.82 percent and 18.28 percent, respectively.

**6.6** 1.95 percent for white females.

## Chapter 7 Answers

**7.1** *a* and *e*.

**7.2** *a*) $x = 5, y = 6, z = 2$. *b*) $x = z = 1, y = -2$.

**7.3** *a*) $x = 17/3, y = -13/3$; *c*) $x = 1, y = -1, z = -2$.

**7.6** *a*) $Y = hI°/(sh + am)$, $r = mI°/(sh + am)$; *b, c*) both decrease as $s$ increases.

**7.8** $x_1 = (b_1a_{22} - b_2a_{12})/(a_{11}a_{22} - a_{12}a_{21})$,
$x_2 = (b_2a_{11} - b_1a_{21})/(a_{11}a_{22} - a_{12}a_{21})$; need $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

**7.10** $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -14 \\ 0 & 1 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & .5 \\ 0 & 1 & .3 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$

**7.12** $w = -1,\ x = 1,\ y = 2,\ z = 3$.

**7.13** *a*) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. *b*) $\begin{pmatrix} 1 & 3 & 4 \\ 0 & -1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

**7.14** $x = (5/4) - (5/4)z$, $y = (3/2) - (3/2)z$.

**7.15** $k = -1$ infinitely many solutions; $k \neq -1$, one solution: $x_1 = 1, x_2 = 0$.

**7.16** *a*) $w, x, y$ basic; $z$ free: $w = (12/11) - (3/11)z$, $x = -(4/11) + (1/11)z$, $y = (7/11) + (12/11)z$; *b*) $w, x$ basic, $y, z$ free: $w = 0.6 - 2.2y + 0.6z$, $x = 0.6 + 0.8y - 0.4z$; *c*) all variables basic; *d*) only one variable basic.

**7.17** *a*) General solution is $x = 1 + (5/4)z$, $y = 12 - (9/4)z$. To get an integer solution, take $z = 4$, then $x = 6$ and $y = 3$. *b*) Just change the right hand side in the system in part *a* and apply the same analysis.

**7.18** $a = -8$.

**7.19** *a*) one, *b*) $q = 0, p = -1$.

**7.20** *a*) 1, *b*) 2, *c*) 3, *e*) 3.

**7.21** *a*) Only the zero solution for *i*, *iii* and *iv*; infinitely many other solutions for the other two. *b*) Unique solution for every RHS for *i* and *iv*; infinitely many solutions for every RHS for *ii*; zero or infinitely many solutions depending on the RHS for *v*; zero or one solution depending on the RHS for *iii*.

**7.23** Only *c*.

**7.25** *i*) : *a*) 2, *b*) $z$ and 1 of the other 3, *c*) $z = 1/4$, $x = (3/4) + w - 2y$.

**7.26** $C = 0.05956 \cdot P$, $S = 0.04702 \cdot P$, $F = 0.35737 \cdot P$.

**7.29** *a*) $z$ and any two of the other 3 can be endogenous; *b*) if $y$ is chosen as exogenous and set to 0, $w = 0.6$, $x = 0.6$, $z = 0$; *c*) if $z$ is the only exogenous variable and set equal to 0, the corresponding system has infinitely many solutions.

**7.30** No, the $3 \times 4$ coefficient matrix has rank 2; no submatrix can have rank 3.

## Chapter 8 Answers

**8.1** *a*) $A + B = \begin{pmatrix} 2 & 4 & 0 \\ 4 & -2 & 4 \end{pmatrix}$, $A - D$ undefined, $3B = \begin{pmatrix} 0 & 3 & -3 \\ 12 & -3 & 6 \end{pmatrix}$,

$$DC = \begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix}, \; B^T = \begin{pmatrix} 0 & 4 \\ 1 & -1 \\ -1 & 2 \end{pmatrix}, \; A^T C^T = \begin{pmatrix} 2 & 6 \\ 1 & 10 \\ 5 & 1 \end{pmatrix}.$$

**8.1** *c*) $CD = \begin{pmatrix} 4 & 3 \\ 5 & 2 \end{pmatrix}$, $DC = \begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix}$.

**8.5** *a*) $AB = \begin{pmatrix} 2 & -5 \\ -5 & 2 \end{pmatrix} = BA$.

**8.7** $\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}$.

$$\begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}.$$

**8.9** number of ways of permuting $n$ objects $= n!$.

**8.10** *a*) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, *not* a permutation matrix; so not closed under addition. *b*) yes, closed under multiplication.

**8.15** Carry out the multiplication.