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# LECTURE 3

# FK1003

EIVIND ERIKSEN

AUG 08TH '14

LINEAR ALGEBRA

## Determinants

$A \rightsquigarrow \det(A) = |A|$

$n\times n$-matrix (square) (a number)

Ex: $n=2$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\det(A) = ad - bc$

$$\boxed{\begin{array}{l}\text{Key property of determinant:} \\ \det(A) \neq 0 \iff A \text{ is invertible}\end{array}}$$

Ex: $(n=2)$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ : $\begin{cases} \text{if } ad-bc \neq 0, \text{ then } A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ \text{if } ad-bc = 0, \text{ then } A^{-1} \text{ does not exist} \end{cases}$

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There is a formula for det(A) when A is any square matrix, but it is complicated and unpractical when $n > 2$.

($n!$ terms)

## 1 Cofactor expansion:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 7 \\ 1 & -1 & 0 \end{pmatrix} \leftarrow$$

choose first row and do cofactor expansion along the first row

$$|A| = 1 \cdot C_{11} + 2 \cdot C_{12} + 3 \cdot C_{13}$$

$$= 1 \cdot (+1) \cdot M_{11} + 2 \cdot (-1) \cdot M_{12} + 3 \cdot (+1) \cdot M_{13}$$

$$= +1 \cdot \begin{vmatrix} 0 & 7 \\ -1 & 0 \end{vmatrix} - 2 \cdot \begin{vmatrix} 4 & 7 \\ 1 & 0 \end{vmatrix} + 3 \cdot \begin{vmatrix} 4 & 0 \\ 1 & -1 \end{vmatrix}$$

---

$C_{ij}$ = cofactor in position $(i,j)$ in A

$= (-1)^{i+j} \cdot M_{ij}$

Sign:
$+ \; - \; + \; - \; + \; \cdots$
$- \; + \; - \; + \; - \; \cdots$
$\vdots$

$M_{ij}$ = minor in position $(i,j)$ in A

= determinant of the matrix obtained from A by deleting row $i$ and column $j$.

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# How to compute determinants:

Two methods: $\begin{cases} \text{- cofactor expension} \\ \text{- Gaussian elimination} \end{cases}$

**Ex:** $n=3$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 7 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 2 \\ 4 & 0 \end{vmatrix} = 1\cdot 0 - 2\cdot 4 = \underline{-8}$$

$$|A| = \left|\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 2 \\ 4 & 0 & 7 & 4 & 0 \\ 1 & -1 & 0 & 1 & -1 \end{array}\right|$$

$$= 1\cdot 0\cdot 0 + 2\cdot 7\cdot 1 + 3\cdot 4\cdot(-1) - 1\cdot 0\cdot 3 - (-1)\cdot 7\cdot 1 - 0\cdot 4\cdot 2$$

$$= 0 + 14 - 12 - 0 + 7 - 0 = \underline{9}$$

* this method only works for $\underline{n=3}$ (not for $n>3$)

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$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 7 \\ 1 & -1 & 0 \end{pmatrix}$$

$$|A| = +1 \cdot \begin{vmatrix} 0 & 7 \\ -1 & 0 \end{vmatrix} - 2 \begin{vmatrix} 4 & 7 \\ 1 & 0 \end{vmatrix} + 3 \cdot \begin{vmatrix} 4 & 0 \\ 1 & -1 \end{vmatrix}$$

$$= 1 \cdot (0 \cdot 0 - 7 \cdot (-1)) - 2(4 \cdot 0 - 7 \cdot 1)$$
$$+ 3 \cdot (4 \cdot (-1) - 1 \cdot 0)$$

$$= 1 \cdot 7 - 2 \cdot (-7) + 3 \cdot (-4)$$
$$= 7 + 14 - 12 = \underline{\underline{9}}$$

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 7 \\ 1 & -1 & 0 \end{pmatrix}$

$$|A| = +1 \quad -(-1)$$

$$= +1 \cdot \begin{vmatrix} 2 & 3 \\ 0 & 7 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 1 & 3 \\ 4 & 7 \end{vmatrix} + \cancel{0 \cdot \begin{vmatrix} & \\ & \end{vmatrix}}$$

$$= 1 \cdot (14) + 1 \cdot (-5) = \underline{9}$$

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**Fact:** Cofactor expansion of A along any row or any column gives the same result.

**Ex:**

$$|A| = \begin{vmatrix} 1 & 7 & 13 & -1 \\ 0 & 4 & 1 & 1 \\ 3 & 1 & 0 & 0 \\ 7 & 2 & -1 & 6 \end{vmatrix}$$

cofactor exp. along third row

$$= +3 \cdot \begin{vmatrix} 7 & 13 & -1 \\ 4 & 1 & 1 \\ 2 & -1 & 6 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 13 & -1 \\ 0 & 1 & 1 \\ 7 & -1 & 6 \end{vmatrix}$$

$$\begin{vmatrix} 13 & -1 \\ -1 & 6 \end{vmatrix}$$

$$= 3 \cdot \left( -4 \cdot 77 + 1 \cdot 44 - 1 \cdot (-33) \right)$$

$$- 1 \left( +1 \cdot 13 - 1(-92) \right)$$

$$= 3 \cdot (-231) - 1 \cdot 105 = -693 - 105$$

$$= -798$$

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# 2 Determinants by Gaussian elimination

Determinant of an echelon form:

Ex:

$$E = \begin{pmatrix} 1 & 7 & 4 \\ 0 & 13 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$|E| = 1 \cdot \begin{vmatrix} 13 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= 1 \cdot (13 \cdot 1)$$

$$= 1 \cdot 13 \cdot 1 = \underline{13}$$

Observation:

i) all entries under the diagonal are zero in an echelon form

(echelon forms are called upper triangular)

ii) the determinant of an upper triangular matrix is the product of the diagonal entries

Ex:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 7 \\ 1 & -1 & 0 \end{pmatrix} \begin{array}{l} -4 \\ \\ -1 \end{array} \qquad |A| = 9$$

$$A' = \begin{pmatrix} 1 & 2 & 3 \\ 0 & \boxed{-8} & \boxed{-5} \\ 0 & \boxed{-3} & \boxed{-3} \end{pmatrix} \begin{array}{l} -3/8 \end{array} \longrightarrow \det(A') = 1 \cdot 9 = \underline{9}$$

$$E = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -8 & -5 \\ 0 & 0 & \frac{15}{8} - 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 0 & -8 & -5 \\ 0 & 0 & -\frac{9}{8} \end{pmatrix} \qquad \det(E) = 1 \cdot (-8) \cdot \left(-\frac{9}{8}\right)$$

$$= +9 = \underline{9}$$

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## Elementary row operations:

(1) Add a multiple ~~of~~ one row to another row

(2) Interchange two rows

(3) Multiply a row with $c \neq 0$.

## Effect on the determinant:

(1) No change

(2) Multiplied $(-1)$

(3) Multiplied $c$

Ex:

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2.$$

$$\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 3 \cdot 2 - 1 \cdot 4 = 2$$

$$\begin{vmatrix} 2 & 4 \\ 3 & 4 \end{vmatrix} = 2 \cdot 4 - 4 \cdot 3 = -4$$

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Ex:

$$\begin{array}{l} \begin{vmatrix} 1 & 7 & 13 & -1 \\ 0 & 4 & 1 & 1 \\ 3 & 1 & 0 & 0 \\ 7 & 2 & -1 & 6 \end{vmatrix} \end{array} \quad -3 \quad -7$$

$$= \begin{vmatrix} 1 & 7 & 13 & -1 \\ 0 & 4 & 1 & 1 \\ 0 & -20 & -39 & 3 \\ 0 & -47 & -92 & 13 \end{vmatrix} = \begin{vmatrix} 4 & 1 & 1 \\ -20 & -39 & 3 \\ -47 & -92 & 13 \end{vmatrix} \quad -3 \quad -13$$

$$= \begin{vmatrix} 4 & 1 & 1 \\ -32 & -42 & 0 \\ -99 & -105 & 0 \end{vmatrix} = \begin{vmatrix} -32 & -42 \\ -99 & -105 \end{vmatrix}$$

$$= 32 \cdot 105 - 42 \cdot 99 = \cancel{3360 - 3301}$$

$$= \cancel{59}$$

$$= 3360 - 4158 = -798$$

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# Applications of the determinant:

## 1 Inverse matrices

Let $A$ be an $n \times n$-matrix. Then

i) If $|A| \neq 0$, then $A$ is invertible and

$$\boxed{A^{-1} = \frac{1}{|A|} \cdot \mathrm{adj}(A)}$$

ii) If $|A| = 0$, then $A$ is not invertible.

**Explanation of adj(A):** $\boxed{\mathrm{adj}(A) = C^T}$

i) Cofactor matrix of $A$: $C = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & & \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}$

ii) Transpose operation:

$A \rightsquigarrow A^T \leftarrow$ the matrix where the rows are colums of $A$

$A$: any $m \times n$ matrix

$A^T$: $n \times m$ matrix

$\operatorname{adj}(A) = C^T$

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 7 \\ 1 & -1 & 0 \end{pmatrix}$ $\qquad A^{-1} = \frac{1}{9} \cdot \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$

$$|A| = +1 \cdot \begin{vmatrix} 0 & 7 \\ -1 & 0 \end{vmatrix} - 2 \cdot \begin{vmatrix} 4 & 7 \\ 1 & 0 \end{vmatrix} + 3 \cdot \begin{vmatrix} 4 & 0 \\ 1 & -1 \end{vmatrix}$$

$$= 1 \cdot 7 - 2 \cdot (-7) + 3 \cdot (-4) = \underline{9} \neq 0$$

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = \underline{\begin{pmatrix} 7 & 7 & -4 \\ -3 & -3 & 3 \\ 14 & 5 & -8 \end{pmatrix}}$$

$C_{11} = 7 \qquad C_{12} = 7 \qquad C_{13} = -4$

$C_{21} = -3 \qquad C_{22} = -3 \qquad C_{23} = -(-3) = 3$

$C_{31} = 14 \qquad C_{32} = 5 \qquad C_{33} = -8$

$$\operatorname{adj}(A) = C^T = \begin{pmatrix} 7 & -3 & 14 \\ 7 & -3 & 5 \\ -4 & 3 & -8 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 7 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\underline{A^{-1} = \frac{1}{9} \cdot \begin{pmatrix} 7 & -3 & 14 \\ 7 & -3 & 5 \\ -4 & 3 & -8 \end{pmatrix}}$$

$A^{-1} \cdot A = I$

$\frac{1}{|A|} \cdot \operatorname{adj}(A) \cdot A = I$

$\operatorname{adj}(A) \cdot A = I \cdot |A|$

② **Kramer's rule:** Method for solving linear systems

$$A \cdot \underline{x} = \underline{b}$$

Ex:

$$x + y + z = 3$$
$$x + 2y + 4z = 7$$
$$x + 3y + 9z = 13$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \cdot \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{= \underline{x}} = \begin{pmatrix} 3 \\ 7 \\ 13 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 3 & 1 & 1 \\ 7 & 2 & 4 \\ 13 & 3 & 9 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 7 & 4 \\ 1 & 13 & 9 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 7 \\ 1 & 3 & 13 \end{pmatrix}$$

**Kramer's rule:**

If $|A| \neq 0$, then $\underline{x} = A^{-1} \cdot \underline{b}$

(one unique solution)

$$x = |A_1| / |A| = 2/2 = 1$$
$$y = |A_2| / |A| = 2/2 = 1$$
$$z = |A_3| / |A| = 2/2 = 1$$

$$|A| = 1 \cdot 6 - 5 + 1 = 2$$
$$|A_1| = 3 \cdot 6 - 11 + (-5) = 2$$
$$|A_2| = 1 \cdot 11 - 3 \cdot 5 + 1 \cdot 6 = 2$$
$$|A_3| = 1 \cdot 5 - 1 \cdot 6 + 3 \cdot 1 = 2$$

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Summary:

After this preparatory course, you should know:

i) How to solve linear systems by Gaussian elimination.

ii) How to compute determinants.

Make sure you do a couple of exercises for i) and ii)!

## 3.1 Introduction to Determinants

*Notation:* $A_{ij}$ is the matrix obtained from matrix $A$ by deleting the $i$th row and $j$th column of $A$.

**EXAMPLE:**

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \qquad A_{23} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Recall that $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ and we let $\det[a] = a$.

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is given by

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + \cdots + (-1)^{1+n}a_{1n}\det A_{1n}$$

$$= \sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det A_{1j}$$

**EXAMPLE:** Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

*Solution*

$$\det A = 1\det\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} - 2\det\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} + 0\det\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$$

$$= \underline{\hspace{8cm}} = \underline{\hspace{1.5cm}}$$

Common notation: $\det\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}$.

So

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix}$$

The **(i, j)-cofactor** of $A$ is the number $C_{ij}$ where
$C_{ij} = (-1)^{i+j}\det A_{ij}$.

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1C_{11} + 2C_{12} + 0C_{13}$$

(cofactor expansion across row 1)

**THEOREM 1** The determinant of an $n \times n$ matrix $A$ can be computed by a cofactor expansion across any row or down any column:

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \qquad \text{(expansion across row } i\text{)}$$

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \qquad \text{(expansion down column } j\text{)}$$

Use a matrix of signs to determine $(-1)^{i+j}$

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

**EXAMPLE:** Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$ using cofactor expansion down column 3.

*Solution*

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 1.$$

**EXAMPLE:** Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{bmatrix}$

*Solution*

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & 1 & 5 \\ 0 & 2 & 1 \\ 0 & 3 & 5 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 & 4 \\ 0 & 2 & 1 \\ 0 & 3 & 5 \end{vmatrix} + 0\begin{vmatrix} 2 & 3 & 4 \\ 2 & 1 & 5 \\ 0 & 3 & 5 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 & 4 \\ 2 & 1 & 5 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= 1 \cdot 2\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 14$$

*Method of cofactor expansion is not practical for large matrices - see Numerical Note on page* 190.

Triangular Matrices:

$$\begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ 0 & 0 & \ddots & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & \ddots & 0 & 0 \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \end{bmatrix}$$

(upper triangular) (lower triangular)

**THEOREM 2**: If $A$ is a triangular matrix, then $\det A$ is the product of the main diagonal entries of $A$.

**EXAMPLE:**

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 4 \end{vmatrix} = \underline{\hspace{6cm}} = -24$$

## 3.2 Properties of Determinants

**THEOREM 3** Let $A$ be a square matrix.

a. If a multiple of one row of $A$ is added to another row of $A$ to produce a matrix $B$, then $\det A = \det B$.

b. If two rows of $A$ are interchanged to produce $B$, then $\det B = -\det A$.

c. If one row of $A$ is multiplied by $k$ to produce $B$, then $\det B = k \cdot \det A$.

**EXAMPLE:** Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix}$.

*Solution*

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix} = 5\begin{vmatrix} 1 & 3 & 4 \\ 2 & 6 & 10 \\ 2 & 7 & 11 \end{vmatrix} = 5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 2 & 7 & 11 \end{vmatrix}$$

$$= 5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = -5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{vmatrix} = \underline{\hspace{3cm}} = \underline{\hspace{1.5cm}}.$$

Theorem 3(c) indicates that $\begin{vmatrix} * & * & * \\ -2k & 5k & 4k \\ * & * & * \end{vmatrix} = k \begin{vmatrix} * & * & * \\ -2 & 5 & 4 \\ * & * & * \end{vmatrix}$.

**EXAMPLE:** Compute $\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix}$

*Solution*

$$\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & -8 & -11 \end{vmatrix}$$

$$= 2(-4)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -8 & -11 \end{vmatrix} = 2(-4)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{vmatrix}$$

$$= 2(-4)(1)(1)(5) = -40$$

**EXAMPLE**: Compute $\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix}$ using a combination of row reduction and cofactor expansion.

*Solution*

$$\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix} = -2\begin{vmatrix} 2 & 3 & 1 \\ 4 & 7 & 3 \\ 1 & 2 & 4 \end{vmatrix} = -2\begin{vmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \\ 0 & 1 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -7 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -7 \\ 0 & 0 & -6 \end{vmatrix} = -2(1)(-1)(-6) = -12.$$

Suppose $A$ has been reduced to $U = \begin{bmatrix} \blacksquare & * & * & \cdots & * \\ 0 & \blacksquare & * & \cdots & * \\ 0 & 0 & \blacksquare & \cdots & * \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix}$ by row replacements and row interchanges, then

$$\det A = \begin{cases} (-1)^r \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

**THEOREM 4** A square matrix is invertible if and only if $\det A \neq 0$.

**THEOREM 5** If $A$ is an $n \times n$ matrix, then $\det A^T = \det A$.

**Partial proof** ($2 \times 2$ case)

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \qquad \text{and}$$

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \det\begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc$$

$$\Rightarrow \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det\begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

(3 × 3 case)

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\det\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = a\begin{vmatrix} e & h \\ f & i \end{vmatrix} - b\begin{vmatrix} d & g \\ f & i \end{vmatrix} + c\begin{vmatrix} d & g \\ e & h \end{vmatrix}$$

$$\Rightarrow \det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \det\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}.$$

*Implications of Theorem 5?*

Theorem 3 still holds if the word *row* is replaced

with ________________________.

**THEOREM 6 (Multiplicative Property)**

For $n \times n$ matrices $A$ and $B$, $\det(AB) = (\det A)(\det B)$.

**EXAMPLE:** Compute $\det A^3$ if $\det A = 5$.

*Solution:* $\det A^3 = \det(AAA) = (\det A)(\det A)(\det A)$

$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_.$

**EXAMPLE:** For $n \times n$ matrices $A$ and $B$, show that $A$ is singular if $\det B \neq 0$ and $\det AB = 0$.

*Solution:* *Since*

$$(\det A)(\det B) = \det AB = 0$$

and

$$\det B \neq 0,$$

then $\det A = 0$. Therefore $A$ is singular.