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# LECTURE 1

# FORK 1003

EIVIND ERIKSEN    AUG 6TH '14    LINEAR ALGEBRA

## Linear Algebra

Lecture I: Linear systems. Gaussian elimination
Lecture 2: Matrices and vectors.
Lecture 3: Determinants. Inverses.

**Notes** //home.bi.no/a0710194
or google "Eivind Eriksen"
} See FK 1003 / GRA 6035

**Textbook:** [ME] Simon, Blume "Math. for economists"
(Part II : Ch 6-11)

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# (1) Linear systems and Gaussian elimination

Ex:

$$\begin{aligned} x+y+z &= 3 \\ x+2y+4z &= 7 \\ x+3y+9z &= 13 \end{aligned}$$

3x3 linear system

In general:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

$m \times n$ linear system $\Big\{$ $m$ = # equations, $n$ = # unknowns (var's)

($a_{ij}, b_i$ : given numbers)

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<u>Gaussian elimination</u> : method for solving linear systems

* efficient method
* pedagogic — makes it easier to understand lin. systems

We are going to learn how to solve relatively small lin. systems with this method.

## Section 1.1: Systems of Linear Equations

A **linear equation:**

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

**EXAMPLE:**

$4x_1 - 5x_2 + 2 = x_1$ and $x_2 = 2(\sqrt{6} - x_1) + x_3$

↓ rearranged ↓

$3x_1 - 5x_2 = -2$ and $2x_1 + x_2 - x_3 = 2\sqrt{6}$

**Not linear:**

$4x_1 - 6x_2 = x_1x_2$ and $x_2 = 2\sqrt{x_1} - 7$

A **system of linear equations** (or a **linear system**):

A collection of one or more linear equations involving the same set of variables, say, $x_1, x_2, \ldots, x_n$.

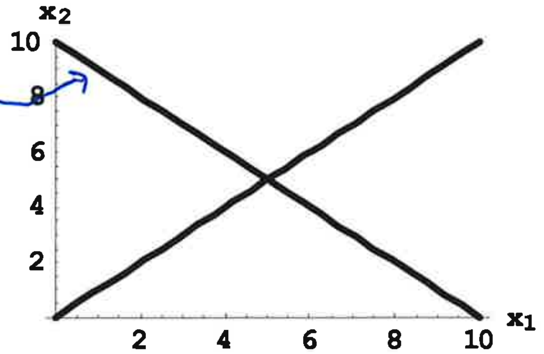
A **solution** of a linear system:

A list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation in the system true when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively.

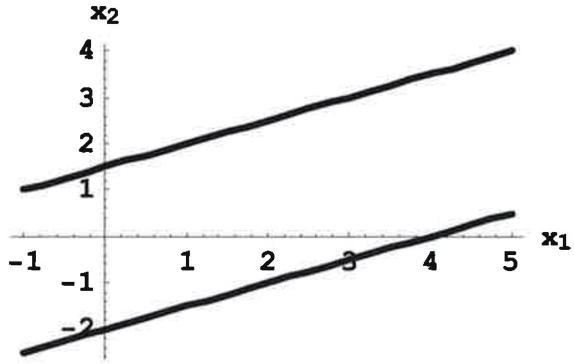
**EXAMPLE** Two equations in two variables:

$x_2 = 10 - x_1$

$$\begin{aligned} x_1 + x_2 &= 10 \\ -x_1 + x_2 &= 0 \end{aligned} \qquad\qquad \begin{aligned} x_1 - 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 8 \end{aligned}$$



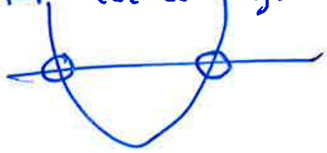

**one unique solution**

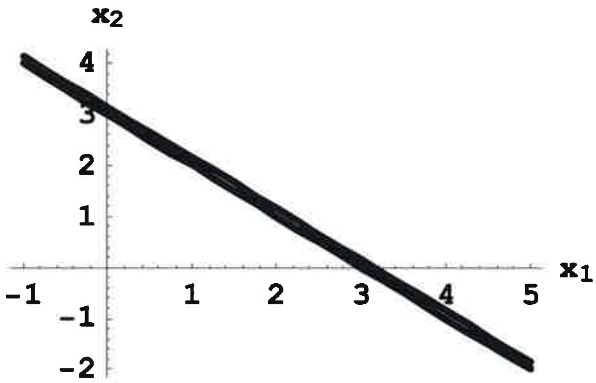



**no solution**

$$\begin{aligned} x_1 + x_2 &= 3 \\ -2x_1 - 2x_2 &= -6 \end{aligned}$$

this cannot happen for linear systems.






**infinitely many solutions**

**BASIC FACT:** A system of linear equations has either

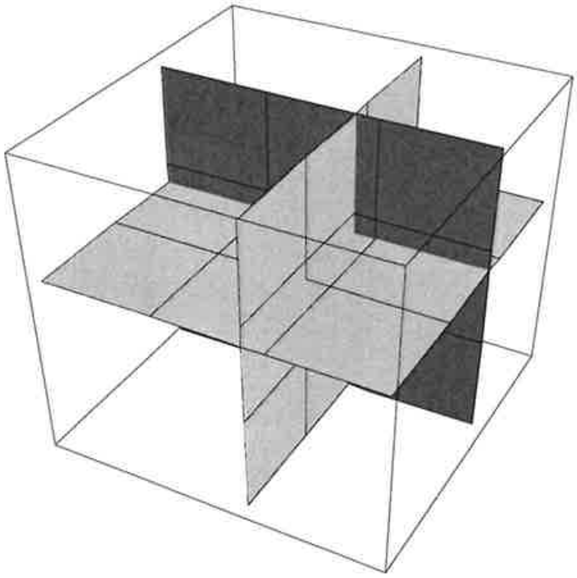
(i) exactly one solution *(consistent)* or

(ii) infinitely many solutions *(consistent)* or

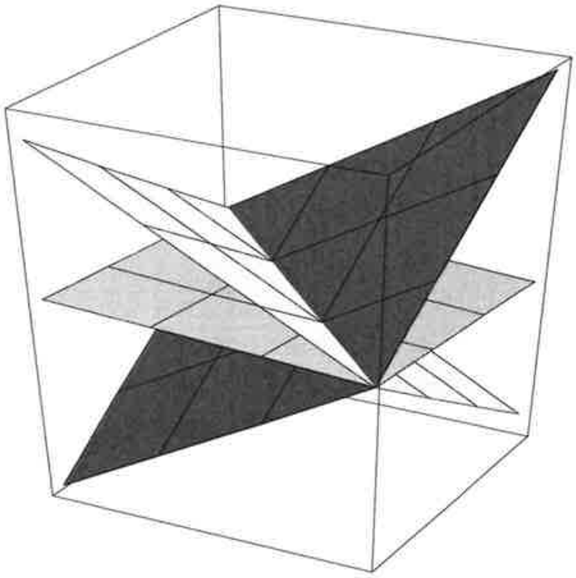
(iii) no solution *(inconsistent)*.

**EXAMPLE:** Three equations in three variables. Each equation determines a plane in 3-space.

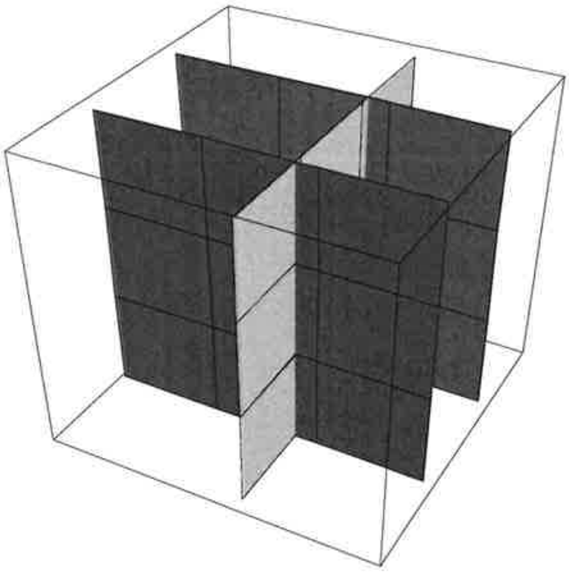
i) The planes intersect in one point. *(one solution)*



ii) The planes intersect in one line. *(infinitely many solutions)*



iii) There is not point in common to all three planes. *(no solution)*



The **solution set:**

- The set of all possible solutions of a linear system.

**Equivalent systems:**

- Two linear systems with the same solution set.

STRATEGY FOR SOLVING A SYSTEM:

- *Replace one system with an equivalent system that is easier to solve.*

**EXAMPLE:**

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ x_2 &= 2 \end{aligned}$$

$$\begin{aligned} x_1 &= 3 \\ x_2 &= 2 \end{aligned}$$

elimination

$-2x_2 + 3x_2 = -1 + 3$

$x_2 = 2$

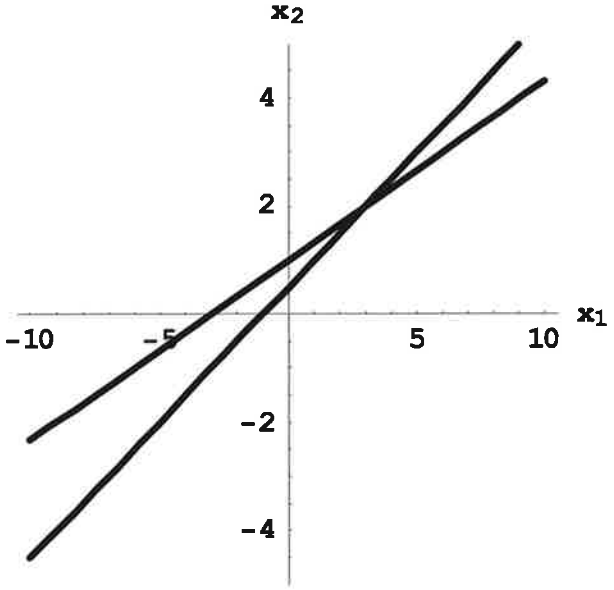
$x_1 = 2x_2 - 1$

$\Downarrow$

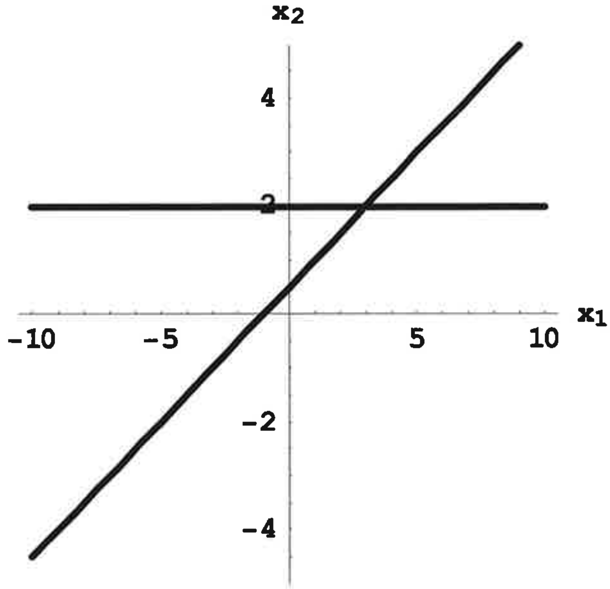
$-(2x_2 - 1) + 3x_2 = 3$

$x_2 = 2$

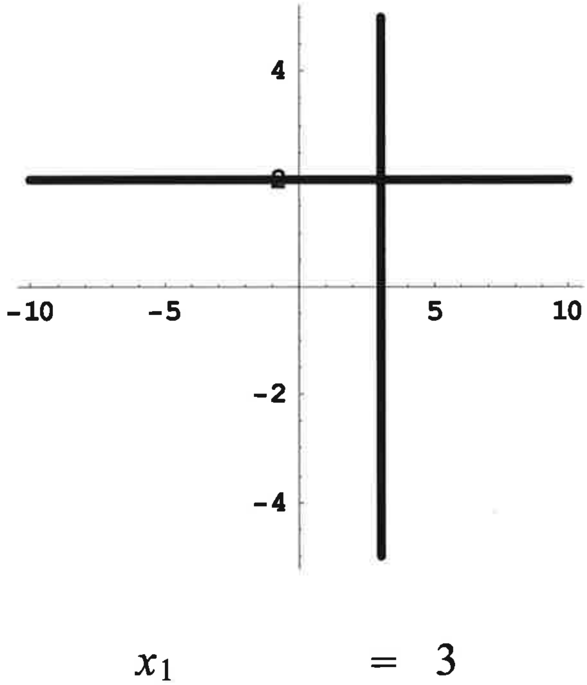
substitution




$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array}$$




$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ & & x_2 & = & 2 \end{array}$$




$$\begin{array}{rcrcr} x_1 & & & = & 3 \\ & & x_2 & = & 2 \end{array}$$

## Matrix Notation

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

(coefficient matrix)

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \qquad \left[\begin{array}{rr|r} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right]$$

(augmented matrix)

↓

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ & & x_2 & = & 2 \end{array} \qquad \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

↓

$$\begin{array}{rcrcr} x_1 & & & = & 3 \\ & & x_2 & = & 2 \end{array} \qquad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

**Elementary Row Operations:**

1. *(Replacement)* Add one row to a multiple of another row.
2. *(Interchange)* Interchange two rows.
3. *(Scaling)* Multiply all entries in a row by a nonzero constant.

**Row equivalent matrices:** Two matrices where one matrix can be transformed into the other matrix by a sequence of elementary row operations.

**Fact about Row Equivalence:** If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

(1) $x + y + z = 3$
(2) $x + 2y + 4z = 7$

$\downarrow$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \end{array}\right) \quad -1$$

$\downarrow$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \end{array}\right)$$

$x + y + z = 3$ (1)
$y + 3z = 4$ $(2) + (-1)\cdot(1)$

**EXAMPLE:**

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \rightarrow \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \quad 4\cdot$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \quad \left[\begin{array}{rrrr} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \quad :2$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \quad \left[\begin{array}{rrrr} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right] \quad 3\cdot$$

$x_1 = 29$

$x_2 = 4\cdot 3 + 4 = 16$

$x_3 = 3$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \leftarrow \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -3 \\ & & x_2 & = & 16 \\ & & x_3 & = & 3 \end{array} \quad \left[\begin{array}{rrrr} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{rcrcrcl} x_1 & & & & & = & 29 \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

**Solution:** $(29, 16, 3)$

**Check:** Is $(29, 16, 3)$ a solution of the *original* system?

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

$$\begin{array}{rcl} (29) - 2(16) + \quad 3 & = 29 - 32 + 3 & = 0 \\ 2(16) - 8(3) & = 32 - 24 & = 8 \\ -4(29) + 5(16) + 9(3) & = -116 + 80 + 27 & = -9 \end{array}$$

**Two Fundamental Questions (Existence and Uniqueness)**

1) Is the system consistent; (i.e. does a solution **exist**?)

2) If a solution exists, is it **unique**? (i.e. is there one & only one solution?)

**EXAMPLE:** Is this system consistent?

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

In the last example, this system was reduced to the triangular form:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This is sufficient to see that the system is consistent and unique. Why?

**EXAMPLE:** Is this system consistent?

$$\begin{aligned} 3x_2 - 6x_3 &= 8 \\ x_1 - 2x_2 + 3x_3 &= -1 \\ 5x_1 - 7x_2 + 9x_3 &= 0 \end{aligned} \qquad \begin{bmatrix} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{bmatrix}$$

**Solution:** Row operations produce:

$$\begin{bmatrix} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 3 & -6 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Equation notation of triangular form:

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -1 \\ 3x_2 - 6x_3 &= 8 \\ 0x_3 &= -3 \quad \leftarrow \textit{Never true} \end{aligned}$$

The original system is inconsistent!

**EXAMPLE:** For what values of $h$ will the following system be consistent?

$$\begin{aligned} 3x_1 &- 9x_2 = 4 \\ -2x_1 &+ 6x_2 = h \end{aligned}$$

**Solution:** Reduce to triangular form.

$$\begin{bmatrix} 3 & -9 & 4 \\ -2 & 6 & h \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & \frac{4}{3} \\ -2 & 6 & h \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & \frac{4}{3} \\ 0 & 0 & h+\frac{8}{3} \end{bmatrix}$$

The second equation is $0x_1 + 0x_2 = h + \frac{8}{3}$. System is consistent only if $h + \frac{8}{3} = 0$ or $h = \frac{-8}{3}$.

Leading ~~term~~ entry = first non-zero entry in a row.

## Section 1.2: Row Reduction and Echelon Forms

**Echelon form** (or **row echelon form**): "trappeform"

1. All nonzero rows are above any rows of all zeros.
2. Each *leading entry* (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

**EXAMPLE:** Echelon forms

(a) $\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \end{bmatrix}$

Check for echelon form:
① Mark all leading terms
② Check that there are only zeros under each leading term

**Reduced echelon form:** Add the following conditions to conditions 1, 2, and 3 above:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

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**Ex:**

$$\begin{aligned} x + y + z &= 3 \\ x + 2y + 4z &= 7 \\ x + 3y + 9z &= 13 \end{aligned} \quad \rightarrow \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right) \begin{array}{l} \\ \leftarrow -1 \\ \leftarrow -1 \end{array}$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right) \begin{array}{l} \\ \\ \leftarrow -2 \end{array}$$

Gaussian elimination

$$\downarrow$$

echelon form

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right) \quad :2$$

$$\leftarrow \quad \begin{aligned} x + y + z &= 3 \\ y + 3z &= 4 \\ 2z &= 2 \end{aligned}$$

**Back substitution**

$2z = 2 \Rightarrow \underline{z = 1}$

$y + 3 = 4 \Rightarrow \underline{y = 1}$

$x + 1 + 1 = 3 \Rightarrow \underline{x = 1}$

**Pivots** = leading coeff. in the echelon form

Gauss-Jordan elim.

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right) \begin{array}{l} \uparrow -1 \\ \uparrow -3 \\ \end{array}$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \begin{array}{l} \uparrow -1 \\ \\ \end{array}$$

$$\downarrow$$

reduced echelon form

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$x = 1, \quad y = 1, \quad z = 1$$

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**Fact:**

- Any augmented matrix is row equivalent to an echelon form (that is, you can get to an echelon form using elementary row operations).
- The echelon form is not unique, but the pivot positions are.
- The reduced echelon form is unique

**EXAMPLE** (continued):

Reduced echelon form :

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

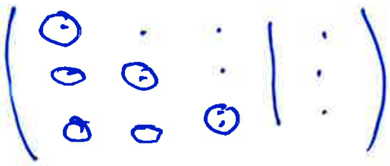
**Theorem 1 (Uniqueness of The Reduced Echelon Form):**

Each matrix is row-equivalent to one and only one reduced echelon matrix.

**Fact:**

You can decide if a system has one, infinitely many or no solutions by considering the pivot positions.

**Ex:** x y z



One solution

no solution

Last eqn: $0x + 0y + 0z = *$ ← nonzero number.

infinitely many solutions

no solutions

$\Updownarrow$

pivot position in the last column of the echelon form.

**Important Terms:**

- **pivot position:** a position of a leading entry in an echelon form of the matrix.

- **pivot:** a nonzero number that either is used in a pivot position to create 0's or is changed into a leading 1, which in turn is used to create 0's.

- **pivot column:** a column that contains a pivot position.

(See the Glossary at the back of the textbook.)

**EXAMPLE:** Row reduce to echelon form and locate the pivot columns.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Switch to get pivot in first col.

**Solution**

pivot ↙

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

↑ pivot column

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Possible Pivots:

echelon form

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \sim \left[\begin{array}{cccc|c} \textcircled{1} & 4 & 5 & -9 & -7 \\ 0 & \textcircled{2} & 4 & -6 & -6 \\ 0 & 0 & 0 & \textcircled{-5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Original Matrix: $\left[\begin{array}{cccc|c} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{array}\right]$

$\uparrow \quad \uparrow \qquad \uparrow$

pivot columns: ① ② ④

not pivot column: ③

**Note:** There is no more than one pivot in any row. There is no more than one pivot in any column.

- infinitely many solutions

Variables: $x_1 \; x_2 \; x_3 \; x_4$

Basic: $x_1 \; x_2 \; x_4$

Free: $x_3$

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$$\left(\begin{array}{cccc|c} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1, x_2, x_4$ : basic
$x_3$ : free

$$x_1 + 4x_2 + 5x_3 - 9x_4 = -7$$
$$2x_2 + 4x_3 - 6x_4 = -6$$
$$-5x_4 = 0$$

$-5x_4 = 0 \Rightarrow x_4 = 0$

$2x_2 + 4x_3 - 6 \cdot 0 = -6$

$\Rightarrow 2x_2 = -6 - 4x_3 \Rightarrow x_2 = -3 - 2x_3$

$x_1 + 4x_2 + 5x_3 - 9x_4 = -7$

$x_1 + 4 \cdot (-3 - 2x_3) + 5x_3 - 9 \cdot 0 = -7$

$x_1 = -7 + 12 + 3x_3$

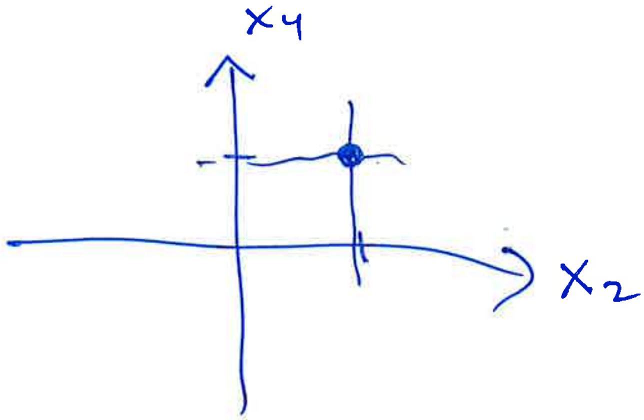
$x_1 = 3x_3 + 5$

$x_1 = 3x_3 + 5$
$x_2 = -2x_3 - 3$
$x_3 = x_3$
$x_4 = 0$

$$\left.\begin{aligned} x_1 &= 3x_3 + 5 \\ x_2 &= -2x_3 - 3 \\ x_3 &= x_3 \\ x_4 &= 0 \end{aligned}\right\} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3x_3+5 \\ -2x_3-3 \\ x_3 \\ 0 \end{pmatrix}$$

$$= x_3 \cdot \begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ -3 \\ 0 \\ 0 \end{pmatrix}$$

\# free variables = degrees of freedom
= dimension of the set of solution




$x_2, x_4$ free

**EXAMPLE:** Row reduce to echelon form and then to reduced echelon form:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

**Solution:**

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Cover the top row and look at the remaining two rows for the left-most nonzero column.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \text{(echelon form)}$$

**Final step to create the reduced echelon form:**
Beginning with the rightmost leading entry, and working upwards to the left, create zeros above each leading entry and scale rows to transform each leading entry into 1.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

## SOLUTIONS OF LINEAR SYSTEMS

- **basic variable:** any variable that corresponds to a pivot column in the augmented matrix of a system.
- **free variable:** all nonbasic variables.

**EXAMPLE:**

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\begin{aligned} x_1 + 6x_2 \quad + 3x_4 \quad &= 0 \\ x_3 - 8x_4 \quad &= 5 \\ x_5 &= 7 \end{aligned}$$

pivot columns:

basic variables:

free variables:

**Final Step in Solving a Consistent Linear System:** After the augmented matrix is in **reduced** echelon form and the system is written down as a set of equations:

*Solve each equation for the basic variable in terms of the free variables (if any) in the equation.*

**EXAMPLE:**

$$\begin{array}{ccccccc} x_1 & +6x_2 & & +3x_4 & & = 0 \\ & & x_3 & -8x_4 & & = 5 \\ & & & & x_5 & = 7 \end{array}$$

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 8x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$$

(general solution)

The **general solution** of the system provides a parametric description of the solution set. (The free variables act as parameters.) The above system has **infinitely many solutions.**

Why?

**Warning: Use only the reduced echelon form to solve a system.**

**Existence and Uniqueness Questions**

**EXAMPLE:**

$$\begin{bmatrix} & 3x_2 & -6x_3 & +6x_4 & +4x_5 & = -5 \\ 3x_1 & -7x_2 & +8x_3 & -5x_4 & +8x_5 & = 9 \\ 3x_1 & -9x_2 & +12x_3 & -9x_4 & +6x_5 & = 15 \end{bmatrix}$$

In an earlier example, we obtained the echelon form:

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad (x_5 = 4)$$

No equation of the form $0 = c$, where $c \neq 0$, so the system is consistent.

**Free variables:** $x_3$ and $x_4$

| **Consistent system with free variables** | $\Rightarrow$ **infinitely many solutions.** |
|---|---|

**EXAMPLE:**

$$\begin{array}{rrcr} 3x_1 & +4x_2 & = & -3 \\ 2x_1 & +5x_2 & = & 5 \\ -2x_1 & -3x_2 & = & 1 \end{array} \rightarrow \begin{bmatrix} 3 & 4 & -3 \\ 2 & 5 & 5 \\ -2 & -3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 4 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{l} 3x_1 + 4x_2 = -3 \\ x_2 = 3 \end{array}$$

| Consistent system, no free variables | $\Rightarrow$ **unique solution.** |
|---|---|

**Theorem 2 (Existence and Uniqueness Theorem)**

1. A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column, i.e., if and only if an echelon form of the augmented matrix has no row of the form

$$\begin{bmatrix} 0 & \dots & 0 & b \end{bmatrix} \text{ (where } b \text{ is nonzero).}$$

2. If a linear system is consistent, then the solution contains either
(i) a unique solution (when there are no free variables) or
(ii) infinitely many solutions (when there is at least one free variable).

Using Row Reduction to Solve Linear Systems

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If not, stop; otherwise go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. State the solution by expressing each basic variable in terms of the free variables and declare the free variables.

**EXAMPLE:**

a) What is the largest possible number of pivots a $4 \times 6$ matrix can have? Why?

b) What is the largest possible number of pivots a $6 \times 4$ matrix can have? Why?

c) How many solutions does a consistent linear system of 3 equations and 4 unknowns have? Why?

d) Suppose the coefficient matrix corresponding to a linear system is $4 \times 6$ and has 3 pivot columns. How many pivot columns does the augmented matrix have if the linear system is inconsistent?