LECTURE 3

FK 1003

EIVIND ERIKSEN AUG 9TH 2015 LINEAR ALGEBRA

Plan:

1. Determinants
2. Inverse matrices

Relevant sections in textbook:
[ME] 9.1 - 9.2,
26.1 - 26.3

Determinant

$A \rightsquigarrow \det(A) = |A|$

A: $n \times n$ matrix; $\det(A)$: number

$n = 2$:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$n > 2$:
more complicated
several methods

Inverse matrices:

A ($n \times n$ matrix): If there is another matrix B such that

$$A \cdot B = B \cdot A = I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

then $B = A^{-1}$ is called the inverse of A.

Ex: $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

Fact: Let A be an $n \times n$-matrix.
i) If $|A| = 0$, then A has no inverse
ii) If $|A| \neq 0$, then A^{-1} exists (and is unique).

$\underline{n=2:}$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $\qquad |A| = ad - bc$

$$A^{-1} = \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{if } ad-bc \neq 0$$

A^{-1} does not exist if $ad - bc = 0$

$\underline{\text{Ex:}}$

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\qquad |A| = 1 \cdot 4 - 3 \cdot 2 = -2 \neq 0$

$$A^{-1} = \frac{1}{-2} \cdot \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}}}$$

$\underline{n>2:}$ $\quad A^{-1}$ exists (uniquely) if $|A| \neq 0$

A^{-1} does not exist if $|A| = 0$

more complicated to find A^{-1}.

Methods for computing $|A|$:

1. Cofactor expansion
2. Gaussian reduction
3. For $n=3$: cross-multiplication

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③ Ex:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = \left|\begin{matrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{matrix}\right| \begin{matrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{matrix}$$

$$= 1\cdot 2\cdot 9 + 1\cdot 4\cdot 1 + 1\cdot 1\cdot 3 - 1\cdot 2\cdot 1 - 3\cdot 4\cdot 1 - 9\cdot 1\cdot 1$$

$$= 18 + 4 + 3 - 2 - 12 - 9 = \underline{\underline{2}}$$

① Cofactor expansion:

Ex:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 1\cdot C_{11} + 1\cdot C_{21} + 1\cdot C_{31}$$

$$= 1\cdot(+1)\cdot\begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} + 1\cdot(-1)\cdot\begin{vmatrix} 1 & 1 \\ 3 & 9 \end{vmatrix} + 1\cdot(+1)\cdot\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix}$$

$$= (2\cdot 9 - 3\cdot 4) - (\overset{9-3}{\cancel{1\cdot 9 - 1\cdot 4}}) + (\overset{4\ -\ 2}{1\cdot 3 - 1\cdot 2})$$

$$= 6 - 6 + 2 = \underline{\underline{2}}$$

3.1 Introduction to Determinants

Notation: A_{ij} is the matrix obtained from matrix A by deleting the ith row and jth column of A.

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 & \cancel{3} & 4 \\ \cancel{5} & \cancel{6} & \cancel{7} & \cancel{8} \\ 9 & 10 & \cancel{11} & 12 \\ 13 & 14 & \cancel{15} & 16 \end{bmatrix} \qquad A_{23} = \begin{bmatrix} 1 & 2 & 4 \\ 9 & 10 & 12 \\ 13 & 14 & 16 \end{bmatrix}$$

Recall that $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ and we let $\det[a] = a$.

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is given by

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + \cdots + (-1)^{1+n}a_{1n}\det A_{1n}$$

$$= \sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det A_{1j}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & & \vdots \end{pmatrix}$$

EXAMPLE: Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

Solution

$$\det A = 1 \det\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} - 2\det\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} + 0\det\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$$

$$= 1\cdot(-1-0) - 2\cdot(3-4) = -1+2 = 1$$

Common notation: $\det\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}$.

So

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix}$$

The **(i,j)-cofactor** of A is the number C_{ij} where $C_{ij} = (-1)^{i+j}\det A_{ij}$.

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1C_{11} + 2C_{12} + 0C_{13}$$

(cofactor expansion across row 1)

THEOREM 1 The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column:

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$ (expansion across row i)

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$ (expansion down column j)

Use a matrix of signs to determine $(-1)^{i+j}$

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

EXAMPLE: Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$ using cofactor expansion down column 3.

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 1.$$

$n=2$: 2 terms
$n=3$: 6 terms
$n=4$: 24 terms
$n=5$: 120 terms

EXAMPLE: Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{bmatrix}$

$|A| = +1 \cdot \begin{vmatrix} 2 & 1 & 5 \\ 0 & 2 & 1 \\ 0 & 3 & 5 \end{vmatrix} = +2 \cdot \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix}$
$= 2(2 \cdot 5 - 3 \cdot 1) = \underline{\underline{14}}$

Solution

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & 1 & 5 \\ 0 & 2 & 1 \\ 0 & 3 & 5 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 & 4 \\ 0 & 2 & 1 \\ 0 & 3 & 5 \end{vmatrix} + 0\begin{vmatrix} 2 & 3 & 4 \\ 2 & 1 & 5 \\ 0 & 3 & 5 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 & 4 \\ 2 & 1 & 5 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= 1 \cdot 2\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 14$$

Method of cofactor expansion is not practical for large matrices - see Numerical Note on page 190.

Triangular Matrices:

$$\begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ 0 & 0 & \ddots & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & \ddots & 0 & 0 \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \end{bmatrix}$$

(upper triangular) (lower triangular)

THEOREM 2: If A is a triangular matrix, then $\det A$ is the product of the main diagonal entries of A.

EXAMPLE:

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 4 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & 5 \\ 0 & 0 & 4 \end{vmatrix} = -24$$

$$= 2 \cdot 1 \cdot \begin{vmatrix} -3 & 5 \\ 0 & 4 \end{vmatrix}$$

$$= 2 \cdot 1 \cdot (-3) \cdot 4 = \underline{\underline{-24}}$$

<u>Gaussian reduction:</u>

$$A \xrightarrow{\text{row operations}} \cdots \longrightarrow E$$

echelon form
↓
upper triangular

3.2 Properties of Determinants

THEOREM 3 Let A be a square matrix.

a. If a multiple of one row of A is added to another row of A to produce a matrix B, then $\det A = \det B$.

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

EXAMPLE: Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix}$.

Solution

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix} = 5\begin{vmatrix} 1 & 3 & 4 \\ 2 & 6 & 10 \\ 2 & 7 & 11 \end{vmatrix} = 5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 2 & 7 & 11 \end{vmatrix}$$

$$= 5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = -5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{vmatrix} = \underline{-5 \cdot 1 \cdot 1 \cdot 2} = \underline{-10}.$$

Ex:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{vmatrix} = 1\cdot(8-6) = \underline{\underline{2}}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{vmatrix} = 1\cdot 1\cdot 2 = \underline{\underline{2}}$$

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 0 & 3 & 0 & 2 \\ 0 & 5 & 1 & 3 \end{vmatrix}$$

$$= 1\cdot \begin{vmatrix} 5 & 0 & 0 \\ 3 & 0 & 2 \\ 5 & 1 & 3 \end{vmatrix} = 5\cdot \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 5\cdot(-2) = \underline{\underline{-10}}$$

Theorem 3(c) indicates that $\begin{vmatrix} * & * & * \\ -2k & 5k & 4k \\ * & * & * \end{vmatrix} = k\begin{vmatrix} * & * & * \\ -2 & 5 & 4 \\ * & * & * \end{vmatrix}$.

EXAMPLE: Compute $\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & -8 & -11 \end{vmatrix}$$

$$= 2(-4)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -8 & -11 \end{vmatrix} = 2(-4)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{vmatrix}$$

$$= 2(-4)(1)(1)(5) = -40$$

EXAMPLE: Compute $\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix}$ using a combination of row reduction and cofactor expansion.

Solution

$$\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix} = -2\begin{vmatrix} 2 & 3 & 1 \\ 4 & 7 & 3 \\ 1 & 2 & 4 \end{vmatrix} = -2\begin{vmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \\ 0 & 1 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -7 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -7 \\ 0 & 0 & -6 \end{vmatrix} = -2(1)(-1)(-6) = -12.$$

Ex:

$$\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix}$$

$$= +(-2)\begin{vmatrix} 2 & 3 & 1 \\ 4 & 7 & 3 \\ 1 & 2 & 4 \end{vmatrix}$$

change sign

$$= 2 \cdot \begin{vmatrix} 1 & 2 & 4 \\ 4 & 7 & 3 \\ 2 & 3 & 1 \end{vmatrix} \begin{array}{l} -4 \\ \\ -2 \end{array}$$

$$= 2 \cdot \begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -13 \\ 0 & -1 & -7 \end{vmatrix} = 2 \cdot 1 \cdot \begin{vmatrix} -1 & -13 \\ -1 & -7 \end{vmatrix}$$

$$= 2 \cdot (7 - 13) = \underline{\underline{-12}}$$

$|A^T| = |A|$:

Ex:
$$\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 7 & 1 \\ 3 & 7 & 9 & 2 \\ 0 & 0 & -2 & 0 \\ 1 & 3 & 4 & 4 \end{vmatrix}$$

Suppose A has been reduced to $U = \begin{bmatrix} \blacksquare & * & * & \cdots & * \\ 0 & \blacksquare & * & \cdots & * \\ 0 & 0 & \blacksquare & \cdots & * \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix}$ by row replacements and row interchanges, then

$$\det A = \begin{cases} (-1)^r \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

THEOREM 4 A square matrix is invertible if and only if $\det A \neq 0$.

THEOREM 5 If A is an $n \times n$ matrix, then $\det A^T = \det A$.

Partial proof (2×2 case)

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \qquad \text{and}$$

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \det\begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc$$

$$\Rightarrow \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det\begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

(3 × 3 case)

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\det\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = a\begin{vmatrix} e & h \\ f & i \end{vmatrix} - b\begin{vmatrix} d & g \\ f & i \end{vmatrix} + c\begin{vmatrix} d & g \\ e & h \end{vmatrix}$$

$$\Rightarrow \det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \det\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}.$$

Implications of Theorem 5?

Theorem 3 still holds if the word *row* is replaced

with column.

THEOREM 6 (Multiplicative Property)

For $n \times n$ matrices A and B, $\det(AB) = (\det A)(\det B)$.

EXAMPLE: Compute $\det A^3$ if $\det A = 5$.

Solution: $\det(A^3) = \det(AAA) = (\det A)(\det A)(\det A)$

$$= \underline{5 \cdot 5 \cdot 5} = \underline{125}.$$

EXAMPLE: For $n \times n$ matrices A and B, show that A is singular if $\det B \neq 0$ and $\det AB = 0$.

(i.e. $|A| \neq 0$)

Solution: *Since*

$$(\det A)(\det B) = \det AB = 0$$

and

$$\det B \neq 0,$$

then $\det A = 0$. Therefore A is singular.

2.2 The Inverse of a Matrix

The inverse of a real number a is denoted by a^{-1}. For example, $7^{-1} = 1/7$ and

$$7 \cdot 7^{-1} = 7^{-1} \cdot 7 = 1$$

An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C satisfying

$$CA = AC = I_n$$

where I_n is the $n \times n$ identity matrix. We call C the **inverse** of A.

FACT If A is invertible, then the inverse is unique.

Proof: Assume B and C are both inverses of A. Then

$$B = BI = B(\underline{AC}) = (\underline{BA})\underline{C} = I\underline{\cdot C} = C.$$

So the inverse is unique since any two inverses coincide.■

The inverse of A is usually denoted by A^{-1}.

We have

$$\boxed{AA^{-1} = A^{-1}A = I_n}$$

Not all $n \times n$ matrices are invertible. A matrix which is *not* invertible is sometimes called a **singular** matrix. An invertible matrix is called **nonsingular** matrix.

Singular : $|A| = 0$

non-singular : $|A| \neq 0$

Theorem 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

Assume A is any invertible matrix and we wish to solve $A\mathbf{x} = \mathbf{b}$. Then

$\underline{A^{-1}\cdot}\, A\mathbf{x} = \underline{A^{-1}\cdot}\,\mathbf{b}$ and so

$I\mathbf{x} = \underline{A^{-1}\cdot b}$ or $\mathbf{x} = \underline{A^{-1}\cdot b}$.

Suppose $\mathbf{w}$ is also a solution to $A\mathbf{x} = \mathbf{b}$. Then $A\mathbf{w} = \mathbf{b}$ and

$\underline{A^{-1}}\, A\mathbf{w} = \underline{A^{-1}}\,\mathbf{b}$ which means $\mathbf{w} = A^{-1}\mathbf{b}$.

So, $\mathbf{w} = A^{-1}\mathbf{b}$, which is in fact the same solution.

We have proved the following result:

Theorem 5

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbf{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

EXAMPLE: Use the inverse of $A = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$ to solve

$$\begin{aligned} -7x_1 &+ 3x_2 &= 2 \\ 5x_1 &- 2x_2 &= 1 \end{aligned}.$$

Solution: Matrix form of the linear system:

$$\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{14-15}\begin{bmatrix} -2 & -3 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}.$$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \end{bmatrix}$$

Theorem 6 Suppose A and B are invertible. Then the following results hold:

a. A^{-1} is invertible and $(A^{-1})^{-1} = A$ (i.e. A is the inverse of A^{-1}).

b. AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

c. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Partial proof of part b:

$$(AB)(B^{-1}A^{-1}) = A(\underline{\quad BB^{-1} \quad})A^{-1}$$

$$= A(\underline{\quad I \quad})A^{-1} = \underline{\quad AA^{-1} \quad} = \underline{\quad I \quad}.$$

Similarly, one can show that $(B^{-1}A^{-1})(AB) = I$.

Theorem 6, part b can be generalized to three or more invertible matrices:

$$(ABC)^{-1} = \underline{\quad C^{-1}B^{-1}A^{-1} \quad}$$

Earlier, we saw a formula for finding the inverse of a 2×2 invertible matrix. How do we find the inverse of an invertible $n \times n$ matrix? To answer this question, we first look at **elementary** matrices.

Elementary Matrices

Definition

An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.

EXAMPLE: Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

E_1, E_2, and E_3 are elementary matrices. Why?

Observe the following products and describe how these products can be obtained by elementary row operations on A.

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{bmatrix}$$

$$E_2A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$$

$$E_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 3a+g & 3b+h & 3c+i \end{bmatrix}$$

If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operations on I_m.

Elementary matrices are *invertible* because row operations are *reversible*. To determine the inverse of an elementary matrix E, determine the elementary row operation needed to transform E back into I and apply this operation to I to find the inverse.

For example,

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

Example: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$. Then

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_2(E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$E_3(E_2 E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So

$$\boxed{E_3 E_2 E_1 A = I_3}.$$

Then multiplying on the right by A^{-1}, we get

$$E_3 E_2 E_1 A \underline{\ A^{-1}\ } = I_3 \underline{\ \cdot A^{-1}\ }.$$

So

$$\boxed{E_3 E_2 E_1 I_3 = A^{-1}}$$

The elementary row operations that row reduce A to $\mathbf{I}_n$ are the same elementary row operations that transform $\mathbf{I}_n$ into $\mathbf{A}^{-1}$.

Theorem 7

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n will also transform I_n to A^{-1}.

Algorithm for finding $\mathbf{A}^{-1}$

Place A and I side-by-side to form an augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$. Then perform row operations on this matrix (which will produce identical operations on A and I). So by Theorem 7:

$$\begin{bmatrix} A & I \end{bmatrix} \text{ will row reduce to } \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

or A is not invertible.

EXAMPLE: Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

Solution:

$$\begin{bmatrix} A & I \end{bmatrix} = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$

Order of multiplication is important!

EXAMPLE Suppose A, B, C, and D are invertible $n \times n$ matrices and $A = B(D - I_n)C$.

Solve for D in terms of A, B, C and D.

Solution:

$$\underline{B^{-1}}\,A\,\underline{C^{-1}} = \underline{B^{-1}}\,B(D - I_n)C\,\underline{\cdot\, C^{-1}}$$

$$D - I_n = B^{-1}AC^{-1}$$

$$D - I_n + \underline{I_n} = B^{-1}AC^{-1} + \underline{I_n}$$

$$D = \underline{B^{-1}AC^{-1} + I_n}$$

How to compute inverse matrices

<u>$n=2$:</u> $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \left\{ \begin{array}{l} A^{-1} = \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad ad-bc \neq 0 \\ \\ \text{and } A^{-1} \text{ does not exist if } ad-bc=0. \end{array} \right.$

<u>$n>2$:</u> Compute $|A|$ and check if $|A| \neq 0$.

<u>Method I:</u> Adjoint formula

If $|A| \neq 0$, then $\boxed{A^{-1} = \frac{1}{|A|} \cdot \text{adj.}(A)}$

where

$$\text{adj}(A) = \begin{pmatrix} C_{11} & C_{12} & C_{13} & \cdots & C_{1n} \\ C_{21} & C_{22} & C_{23} & \cdots & C_{2n} \\ \vdots & & & & \vdots \\ C_{n1} & C_{n2} & C_{n3} & \cdots & C_{nn} \end{pmatrix}^T$$

<u>Ex:</u> $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$ $\quad |A| = 2$

$$C = \begin{pmatrix} 6 & -5 & +1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

$C_{11} = + \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} = 6$

$C_{12} = - \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} = -5$

$C_{13} = + \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1$

$C_{21} = -6 \quad C_{22} = 8 \quad C_{23} = -2$

$C_{31} = 2 \quad C_{32} = -3 \quad C_{33} = 1$

$$A^{-1} = \frac{1}{2} \cdot \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}$$

Method 2: Find A^{-1} using row operations.

$$|A|\neq 0: \left(\begin{array}{c|c} A & I \end{array}\right) \xrightarrow{} \xrightarrow{} \xrightarrow{} \cdots \xrightarrow{} \left(\begin{array}{c|c} I & A^{-1} \end{array}\right)$$

row operation $\qquad$ reduced echelon form.

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$

> <u>Note</u>: If $|A|=0$, the reduced echelon form will have less than n pivot positions

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & 9 & 0 & 0 & 1 \end{array}\right) \begin{matrix} \\ ]\cdot(-1) \\ ]\cdot(-1) \end{matrix} \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 2 & 8 & -1 & 0 & 1 \end{array}\right)]\cdot(-2)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right) \cdot \tfrac{1}{2} \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right) \begin{matrix}]\cdot(-1) \\ ]\cdot(-3) \\ \end{matrix}$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1/2 & 1 & -1/2 \\ 0 & 1 & 0 & -5/2 & 4 & -3/2 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right)]\cdot(-1)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -5/2 & 4 & -3/2 \\ 0 & 0 & 1 & 1/2 & -1 & 1/2 \end{array}\right) = \left(\begin{array}{c|c} I & A^{-1} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -5/2 & 4 & -3/2 \\ 1/2 & -1 & 1/2 \end{pmatrix} = \frac{1}{2} \cdot \underline{\underline{\begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}}}$$

What do we use A^{-1} for?

<u>Ex:</u>

$$\left.\begin{aligned} -7x + 3y &= 2 \\ 5x - 2y &= 1 \end{aligned}\right\}$$

$A \cdot \underline{x} = \underline{b}$
matrix form

$$\begin{pmatrix} -7 & 3 \\ 5 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -7 & 3 \\ 5 & -2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \underbrace{\frac{1}{-1} \cdot \begin{pmatrix} -2 & -3 \\ -5 & -7 \end{pmatrix}} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 17 \end{pmatrix}$$

$x = 7$
$y = 17$

$$A \underline{x} = \underline{b}$$
$$\underbrace{A^{-1} \cdot A}\underline{x} = A^{-1} \cdot \underline{b}$$
$$I \underline{x} = A^{-1} \cdot b$$
$$\underline{x} = A^{-1} \underline{b}$$

Linear systems:

$n \times n$ – quadratic linear system

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned}\right\}$$

$A \cdot \underline{x} = \underline{b}$

matrix form

Facts:

$|A| \neq 0$: A^{-1} exists

$A\underline{x} = \underline{b}$

$\boxed{\underline{x} = A^{-1} \cdot b}$ (one solution)

$|A| = 0$: A^{-1} does not exist

no solutions

or

infinitely many solutions

$$A \xrightarrow{\text{row operation}} \cdots \to E$$

$$\left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{array}\right) \to \to \dots \left(\begin{array}{ccc|c} \boxtimes & * & * & * \\ 0 & \boxtimes & * & * \\ 0 & 0 & 0 & * \end{array}\right)$$

$* = 0$

free var's

inf. many solutions

$* \neq 0$

no solutions

$\det(A) \neq 0$:

n pivots

$\det(A) = 0$:

less than n pivots

Problems:

1) Is $\begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix}$ a linear combination of the vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$?

2) Compute the determinant

$$\begin{vmatrix} 1 & 3 & 7 \\ 2 & 4 & 0 \\ 1 & 3 & 1 \end{vmatrix}$$

3) Compute the determinant

$$\begin{vmatrix} 1 & 4 & 0 & 6 \\ 7 & -1 & 0 & -1 \\ 1 & 0 & 3 & 4 \\ 2 & 4 & 1 & 0 \end{vmatrix}$$

4) Compute

$$\begin{pmatrix} 1 & 2 & 4 \\ 3 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 7 & 4 \end{pmatrix}$$

Solución:

1)
$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 1 & -3 & 1 & -7 \\ 1 & 1 & 2 & 3 \end{array}\right) \begin{matrix} \\ ]\cdot(-1) \\ ]\cdot(-1) \end{matrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -4 & 1 & -9 \\ 0 & 0 & 2 & 1 \end{array}\right)$$

echelon form

Yes, it is a linear combination. $\Leftarrow$ one solution

2)
$$\begin{vmatrix} 1 & 3 & 7 \\ 2 & 4 & 0 \\ 1 & 3 & 1 \end{vmatrix} \begin{matrix}]\cdot(-2) \\ \\ ]\cdot(-1) \end{matrix} = \begin{vmatrix} 1 & 3 & 7 \\ 0 & -2 & -14 \\ 0 & 0 & -6 \end{vmatrix} = 1\cdot(-2)\cdot(-6) = \underline{\underline{12}}$$

3)
$$\begin{vmatrix} 1 & 4 & 0 & 6 \\ 7 & -1 & 0 & -1 \\ 1 & 0 & 3 & 4 \\ 2 & 4 & 1 & 0 \end{vmatrix} \begin{matrix}]\cdot(-7) \\ ]\cdot(-1) \\ ]\cdot(-2) \end{matrix} = \begin{vmatrix} 1 & 4 & 0 & 6 \\ 0 & -29 & 0 & -43 \\ 0 & -4 & 3 & -2 \\ 0 & -4 & 1 & -12 \end{vmatrix}$$

$$= \begin{vmatrix} -29 & 0 & -43 \\ -4 & 3 & -2 \\ -4 & 1 & -12 \end{vmatrix} = -29\cdot(-36+2) - 43(-4+12)$$

$$= +29\cdot 34 - 43\cdot 8 = \underline{\underline{642}}$$

4)
$$\begin{pmatrix} 1 & 2 & 4 \\ 3 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 7 & 4 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 33 & 18 \\ 1 & -1 \\ 7 & 4 \end{pmatrix}}}$$

Main points in this course

1. Linear systems and Gaussian elimination
2. Determinants

Make sure you know how to compute

1. Gaussian elimination
2. Determinants

There is a self-test ("final test") with solutions in It's Learning, and also more exercises.

Questions: $\boxed{\text{Office B4-032}}$