LECTURE 2

FK 1003

LINEAR ALGEBRA

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AUG 8TH 2013

PLAN:

1. Review: Gaussian elimination
2. Vectors, matrices and matrix algebra.
3. Determinants.

Relevant part of textbook:

[MEJ] 8.1–8.4, 9.1–9.2, 10.1–10.3

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1. Review: Gaussian elimination

Linear system

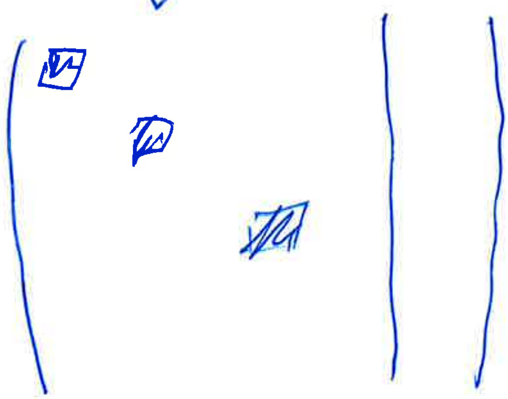
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots &= b_m \end{aligned}$$

→

Augmented matrix

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & \dots & b_1 \\ a_{21} & a_{22} & \dots & b_2 \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & b_m \end{array}\right)$$

↓ Elementary row operations



echelon form

← Write down the equations corresponding to the echelon form, and find solutions of the linear system.

Problems: Solve the linear systems using Gaussian elimination:

1. $$\begin{aligned} x + y + z &= 3 \\ x + 2y + 4z &= 7 \\ x + 3y + 9z &= 13 \end{aligned}$$

2. $$\begin{aligned} x + y + z &= 4 \\ x - y + 2z &= 1 \\ x + 5y - z &= 10 \end{aligned}$$

3. $$\begin{aligned} x + y + z + w &= 4 \\ x - y + z - w &= 1 \\ x + 3y + z + 3w &= 7 \end{aligned}$$

Solutions:

1.) $$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right) \begin{matrix} -1 \\ -1 \end{matrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right) \,]-2$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

$$\begin{aligned} x + y + z &= 3 \\ y + 3z &= 4 \\ 2z &= 2 \end{aligned} \qquad \begin{aligned} x &= 1 \\ y &= 1 \\ z &= 1 \end{aligned}$$

echelon form

unique solution

2.)
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -1 & 2 & 1 \\ 1 & 5 & -1 & 10 \end{array}\right) \begin{matrix} -1 \\ -1 \end{matrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & 1 & -3 \\ 0 & 4 & -2 & 6 \end{array}\right) \, 2$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

echelon form
inf. many solutions
(z free var.)

$x + y + z = 4$
$-2y + z = -3$

$$y = \frac{-3-z}{-2} = \frac{3}{2} + \frac{1}{2}z$$

$$x = 4 - y - z$$
$$= 4 - \left(\frac{3}{2} + \frac{1}{2}z\right) - z$$
$$= \frac{5}{2} - \frac{3}{2}z$$

Solutions:

$x = \frac{5}{2} - \frac{3}{2}z$

$y = \frac{3}{2} + \frac{1}{2}z$

$z =$ free

3).
$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 3 & 1 & 3 & 7 \end{array}\right) \begin{matrix} -1 \\ -1 \end{matrix} \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & -2 & -3 \\ 0 & 2 & 0 & 2 & 3 \end{array}\right) \, 1$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

echelon form
(z, w free)

$x + y + z + w = 4$
$-2y \quad -2w = -3$

$$y = \frac{2w-3}{-2} = -w + 3/2$$

$$x = 4 - (-w + 3/2) - z - w$$
$$= 5/2 - z$$

② Vectors, matrices, matrix algebra

A $m \times n$ <u>matrix</u> is a rectangular array of numbers (m rows, n columns).

A <u>vector</u> (column vector) is a matrix with one column.

<u>Ex:</u>

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad (2\times 2) \qquad a_{11} = 1 \quad a_{12} = 2 \quad a_{21} = 3 \quad a_{22} = 4$$

$$B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 7 & 4 \end{pmatrix} \quad (2\times 3) \qquad b_{13} = -1$$

$$\underline{v} = \vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Matrix operations:

1) <u>Addition / subtraction:</u> $A+B$, $A-B$ (defined when A, B have same dim.)

<u>Ex:</u>

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 7 & -4 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 8 & -2 \\ 6 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 7 & -4 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} -6 & 6 \\ 0 & 4 \end{pmatrix}$$

2) Scalar multiplication

scalar = number

$r \cdot A$ r number (scalar), A matrix

Ex: $2 \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$

3) Matrix multiplication

$A \cdot B \rightsquigarrow$

($m \times n$ matrix)   ($n \times p$ matrix)   ($m \times p$ matrix)

columns in A =
rows in B

$(1 \;\; 2) \cdot \begin{pmatrix} 7 \\ -1 \end{pmatrix} = 1 \cdot 7 + 2 \cdot (-1) = 7 - 2 = \underline{5}$

Ex: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 17 \end{pmatrix}$

(2×2) (2×1) $\longrightarrow$ (2×1)

$(3 \;\; 4) \cdot \begin{pmatrix} 7 \\ -1 \end{pmatrix} = 3 \cdot 7 + 4 \cdot (-1) = 21 - 4 = \underline{17}$

Note: $A \cdot B \neq B \cdot A$

$$\begin{array}{c|c} & \begin{pmatrix} 7 \\ -1 \end{pmatrix} \\ \hline \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} & \begin{pmatrix} 5 \\ 17 \end{pmatrix} \end{array}$$

Ex:

$$\begin{pmatrix} 1 & 0 \\ -1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 3 & 7 \\ 2 & 8 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 3 & 7 \\ 5 & 25 \end{pmatrix}}}$$

	$\begin{pmatrix} 3 & 7 \\ 2 & 8 \end{pmatrix}$
row → $\begin{pmatrix} 1 & 0 \\ -1 & 4 \end{pmatrix}$	$\begin{pmatrix} 3 & 7 \\ 5 & 25 \end{pmatrix}$

4) Transpose:

$A \rightsquigarrow A^T, A^t$

(m×n matrix) (n×m matrix)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

(2×3) (3×2)

1.3 VECTOR EQUATIONS

Key concepts to master: linear combinations of vectors and a spanning set.

Vector: A matrix with only one column.

Vectors in $\mathbf{R}^n$ (vectors with n entries):

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Geometric Description of $\mathbf{R}^2$

Vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the point (x_1, x_2) in the plane.

$\mathbf{R}^2$ is the set of all points in the plane.

Operations for vectors: addition, subtraction, scalar multiplication

Parallelogram rule for addition of two vectors:

If **u** and **v** in $\mathbf{R}^2$ are represented as points in the plane, then $\mathbf{u}+\mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are **0**, **u** and **v**. (Note that $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.)

EXAMPLE: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Graphs of **u**, **v** and $\mathbf{u}+\mathbf{v}$ are given below:

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$\underline{u} + \underline{v}$

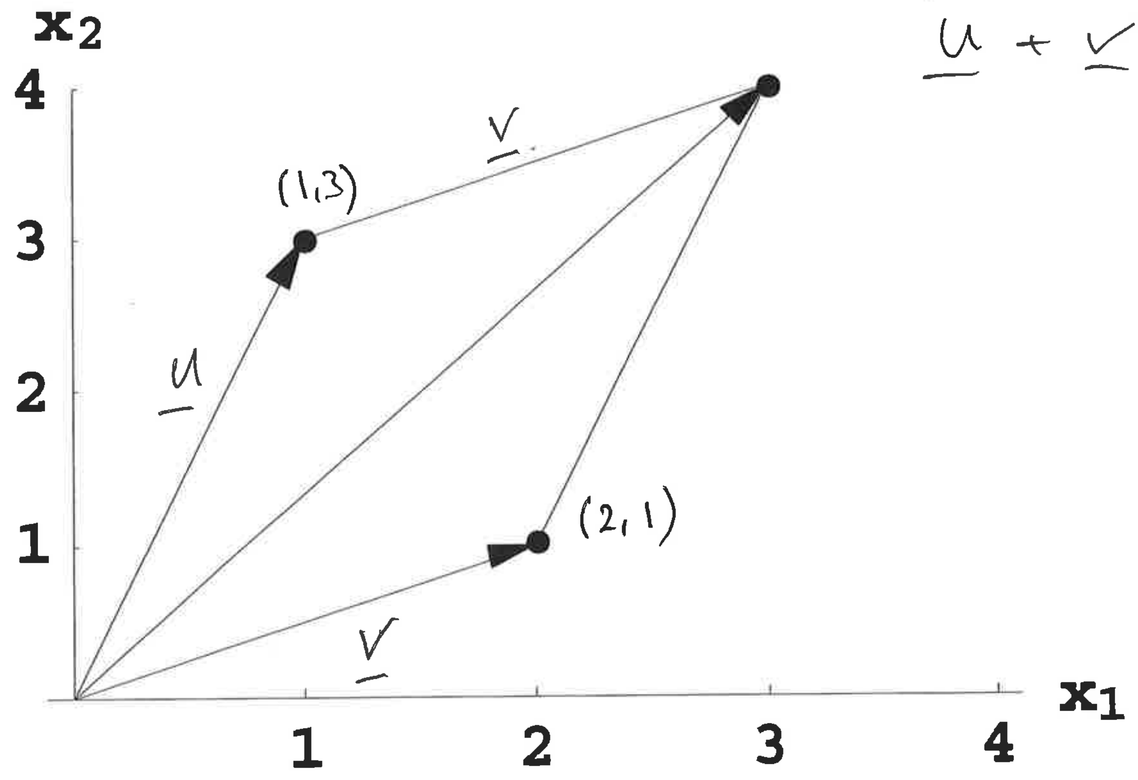


Illustration of the Parallelogram Rule

Linear Combinations

DEFINITION

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbf{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ using weights $c_1, c_2, \ldots, c_p$.

Examples of linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$3\mathbf{v}_1 + 2\mathbf{v}_2, \qquad \tfrac{1}{3}\mathbf{v}_1, \qquad \mathbf{v}_1 - 2\mathbf{v}_2, \qquad \mathbf{0}$$

$\frac{1}{3}\mathbf{v}_1 = \frac{1}{3} v_1 + 0 \cdot v_2$, $\quad \mathbf{0} = 0 \cdot v_1 + 0 \cdot v_2$

Ex:

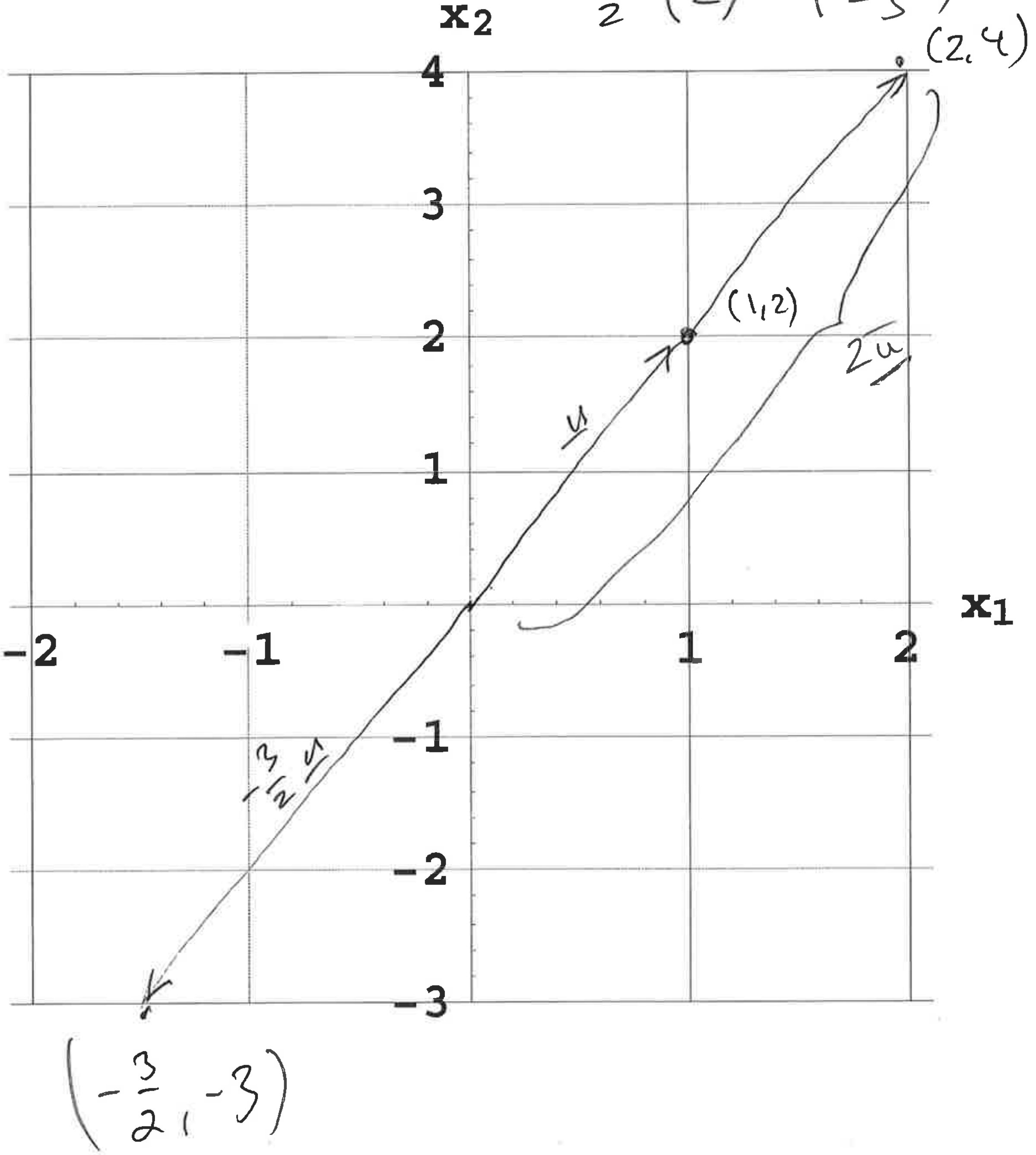
$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$$3v_1 + 2v_2 = 3 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2 \cdot \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} + \begin{pmatrix} 6 \\ -4 \end{pmatrix} = \begin{pmatrix} 12 \\ -1 \end{pmatrix}$$

EXAMPLE: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Express $\mathbf{u}$, $2\mathbf{u}$, and $\frac{-3}{2}\mathbf{u}$ on a graph.

$2 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$-\frac{3}{2} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3/2 \\ -3 \end{pmatrix}$

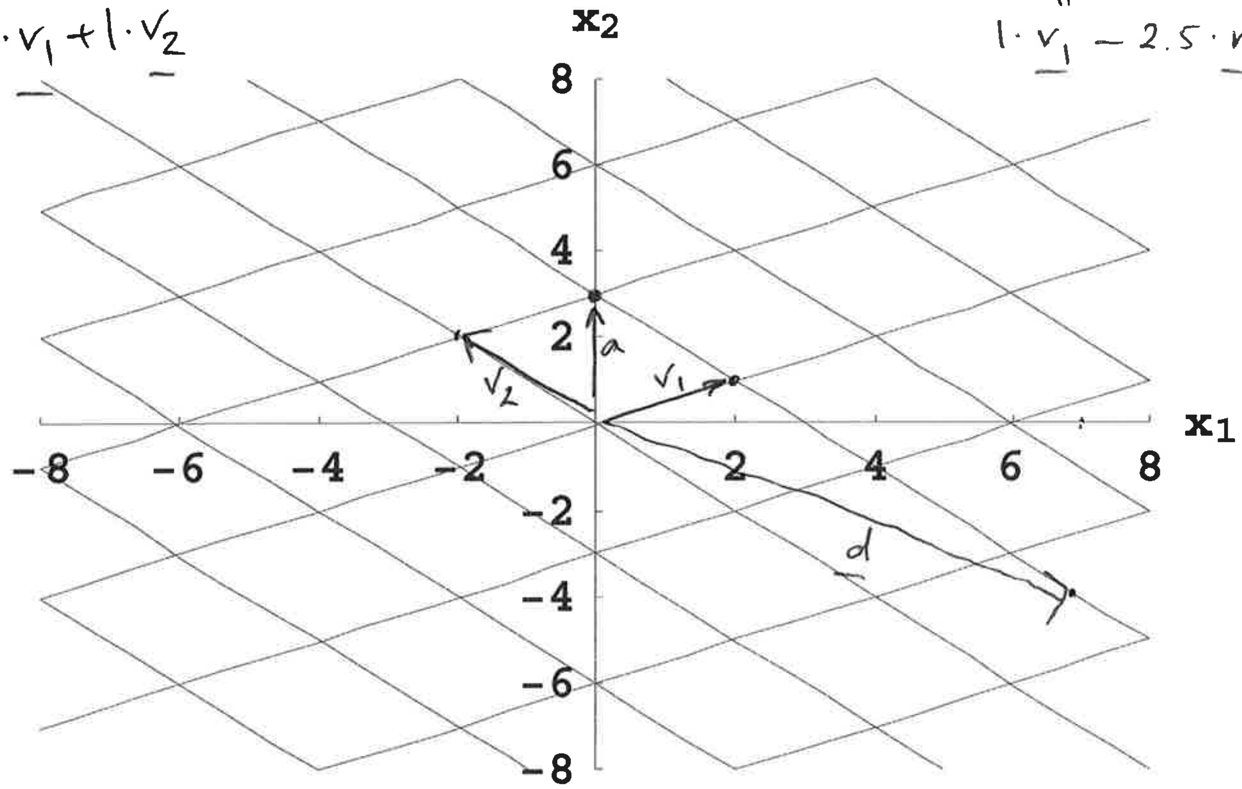


EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Express each of the following as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{a} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$

$= 1 \cdot v_1 + 1 \cdot v_2$ (for **a**)

$= 1 \cdot v_1 - 2.5 \cdot v_2$ (for **d**)



EXAMPLE: Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}$,

and $\mathbf{b} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$.

Are there scalers c_1, c_2, c_3 s.t.

$x_1 \cdot \underline{a}_1 + x_2 \cdot \underline{a}_2 + x_3 \cdot \underline{a}_3 = \underline{b}$

Determine if **b** is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

Solution: Vector **b** is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ if can we find weights x_1, x_2, x_3 such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}.$$

Vector Equation (fill-in):

$$x_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 4 \\ 2 \\ 14 \end{pmatrix} + x_3 \cdot \begin{pmatrix} 3 \\ 6 \\ 10 \end{pmatrix} = \begin{pmatrix} -1 \\ 8 \\ -5 \end{pmatrix}$$

$$\begin{pmatrix} x_1 + 4x_2 + 3x_3 \\ 2x_2 + 6x_3 \\ 3x_1 + 14x_2 + 10x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 8 \\ -5 \end{pmatrix}$$

Corresponding System:

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= -1 \\ 2x_2 + 6x_3 &= 8 \\ 3x_1 + 14x_2 + 10x_3 &= -5 \end{aligned}$$

Corresponding Augmented Matrix:

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = \underline{\ 1\ } \\ x_2 = \underline{-2} \\ x_3 = \underline{\ 2\ } \end{array}$$

augmented matrix

echelon form

$\underline{b} = 1 \cdot \underline{a}_1 - 2\underline{a}_2 + 2\underline{a}_3$

yes, it is a linear combination.

Review of the last example: $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{b}$ are columns of the augmented matrix

$$\begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{bmatrix}$$
$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{array}$$

Solution to

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$$

is found by solving the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{bmatrix}.$$

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

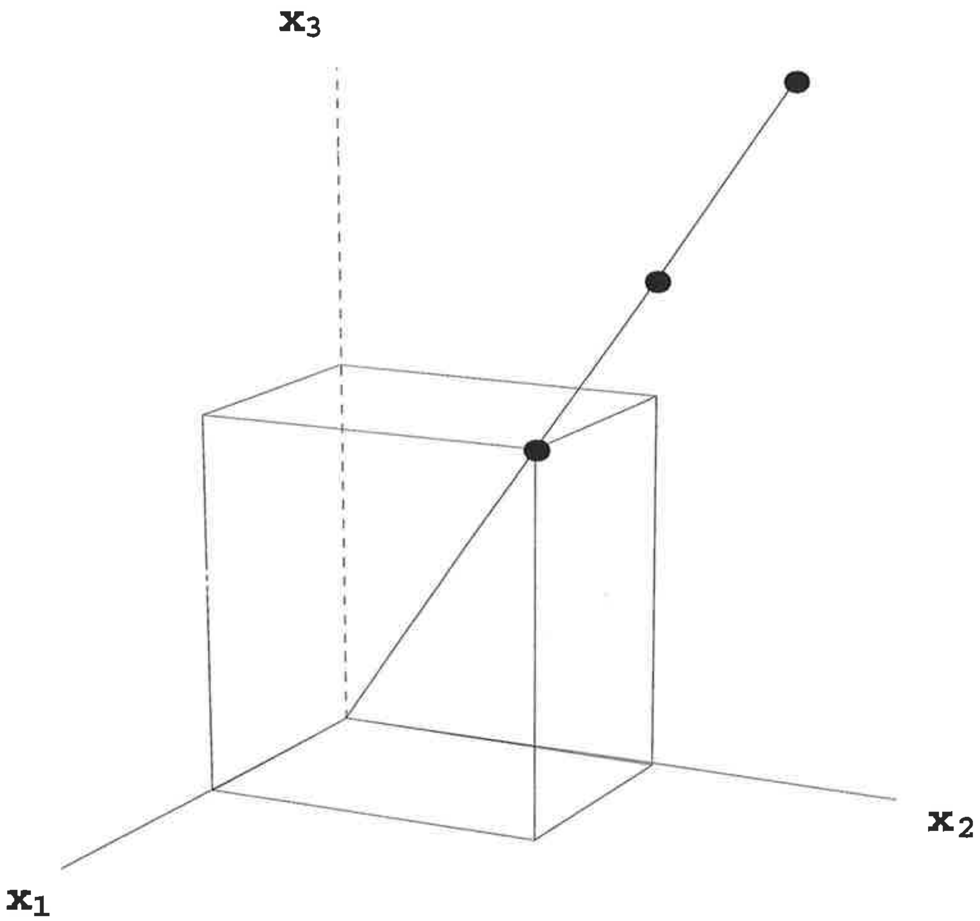
has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}.$$

In particular, **b** can be generated by a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ if and only if there is a solution to the linear system corresponding to the augmented matrix.

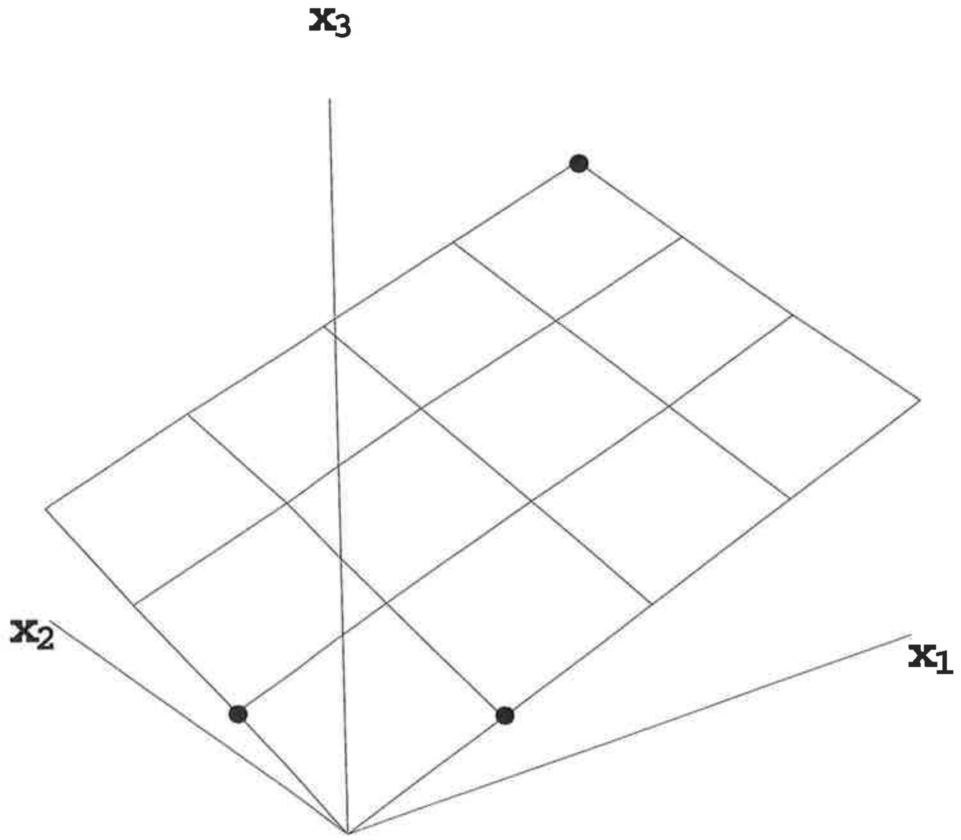
The Span of a Set of Vectors

EXAMPLE: Let $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$. Label the origin $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ together with $\mathbf{v}$, $2\mathbf{v}$ and $1.5\mathbf{v}$ on the graph below.



$\mathbf{v}$, $2\mathbf{v}$ and $1.5\mathbf{v}$ all lie on the same line.
Span$\{\mathbf{v}\}$ is the set of all vectors of the form $c\mathbf{v}$.
Here, **Span**$\{\mathbf{v}\}$ = a line through the origin.

EXAMPLE: Label **u**, **v**, **u** + **v** and 3**u** +4**v** on the graph below.



u, **v**, **u** + **v** and 3**u** +4**v** all lie in the same plane.
Span$\{\mathbf{u},\mathbf{v}\}$ is the set of all vectors of the form $x_1\mathbf{u} + x_2\mathbf{v}$.
Here, **Span**$\{\mathbf{u},\mathbf{v}\}$ = a plane through the origin.

Definition

Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are in $\mathbf{R}^n$; then

Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ = set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$.

Stated another way: **Span**$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written as

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p$$

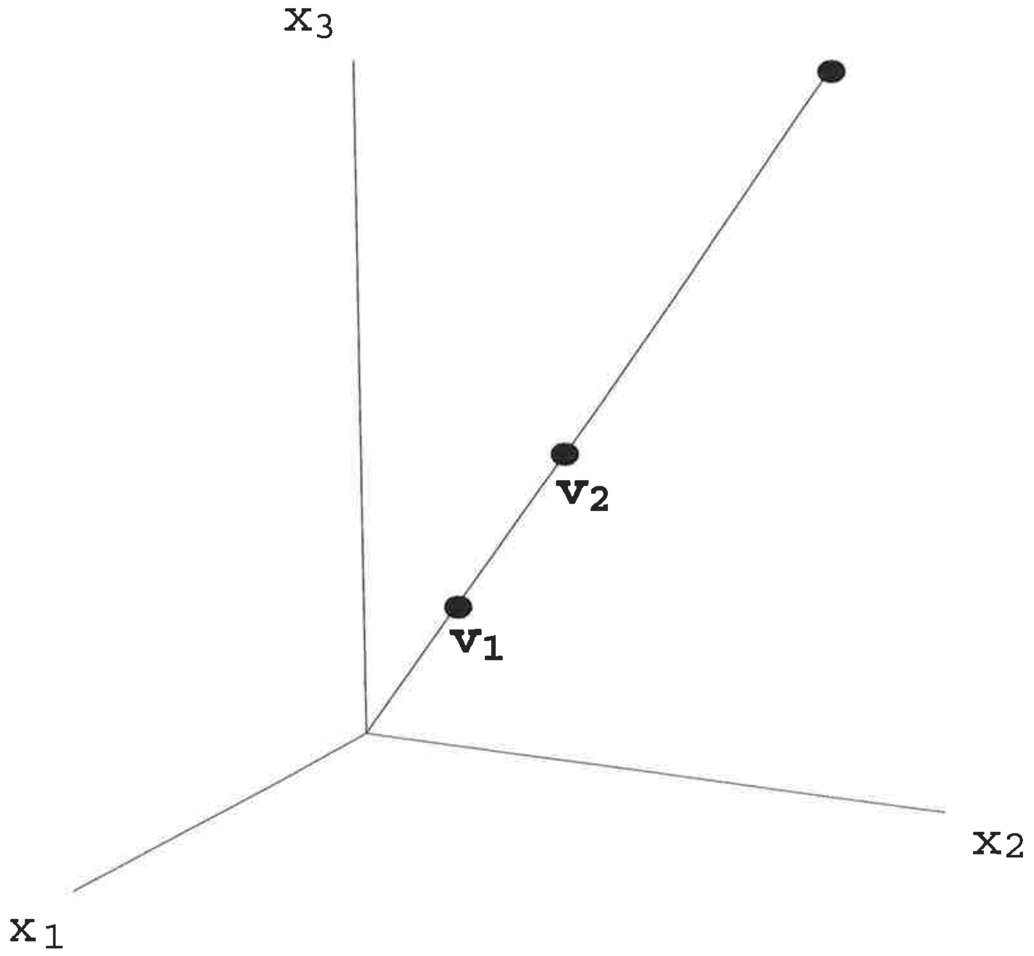
where $x_1, x_2, \ldots, x_p$ are scalars.

EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

(a) Find a vector in **Span**$\{\mathbf{v}_1, \mathbf{v}_2\}$.

(b) Describe **Span**$\{\mathbf{v}_1, \mathbf{v}_2\}$ geometrically.

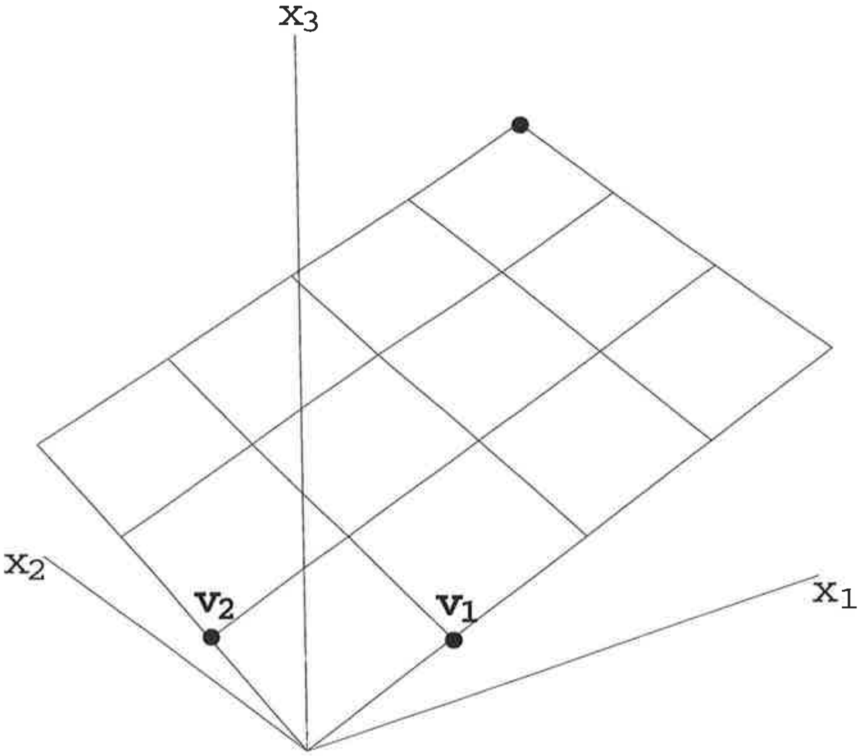
Spanning Sets in $\mathbf{R}^3$



$\mathbf{v}_2$ is a multiple of $\mathbf{v}_1$

$\mathbf{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \mathbf{Span}\{\mathbf{v}_1\} = \mathbf{Span}\{\mathbf{v}_2\}$

(line through the origin)



$\mathbf{v}_2$ is **not** a multiple of $\mathbf{v}_1$

Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ =plane through the origin

EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$. Is **Span**$\{\mathbf{v}_1, \mathbf{v}_2\}$ a line or a plane?

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}$. Is **b** in the plane spanned by the columns of A?

Solution:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}$$

Do x_1 and x_2 exist so that

Corresponding augmented matrix:

$$\begin{bmatrix} 1 & 2 & 8 \\ 3 & 1 & 3 \\ 0 & 5 & 17 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 5 & 17 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 0 & -4 \end{bmatrix}$$

So **b** is not in the plane spanned by the columns of A

Matrix equation of a linear system

Linear system:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

$\Updownarrow$

Vector equation:

$$x_1 \cdot \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \cdot \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \cdot \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$x_1 \underline{a}_1 + x_2 \cdot \underline{a}_2 + \dots + x_n \cdot \underline{a}_n = \underline{b}$$

$\Updownarrow$

Matrix equation:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$A \cdot \underline{x} = \underline{b}$$

Ex:

$$\begin{aligned} x + y + z &= 3 \\ x + 2y + 4z &= 7 \\ x + 3y + 9z &= 13 \end{aligned}$$

$$A \cdot \underline{x} = \underline{b}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 13 \end{pmatrix}$$

1.4 The Matrix Equation Ax = b

Linear combinations can be viewed as a matrix-vector multiplication.

Definition

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbf{R}^n$, then the **product of A and x**, denoted by $A\mathbf{x}$, is the **linear combination of the columns of A using the corresponding entries in x as weights**. I.e.,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\,\mathbf{a}_1 + x_2\,\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

EXAMPLE:

$$\begin{bmatrix} 1 & -4 \\ 3 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ -6 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + -6\begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix} =$$

$$\begin{bmatrix} 7 \\ 21 \\ 0 \end{bmatrix} + \begin{bmatrix} 24 \\ -12 \\ -30 \end{bmatrix} = \begin{bmatrix} 31 \\ 9 \\ -30 \end{bmatrix}$$

EXAMPLE: Write down the system of equations corresponding to the augmented matrix below and then express the system of equations in vector form and finally in the form $A\mathbf{x} = \mathbf{b}$ where **b** is a 3×1 vector.

$$\begin{bmatrix} 2 & 3 & 4 & 9 \\ -3 & 1 & 0 & -2 \end{bmatrix}$$

Solution: Corresponding system of equations (fill-in)

Vector Equation:

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \end{bmatrix}.$$

Matrix equation (fill-in):

Three equivalent ways of viewing a linear system:

1. as a system of linear equations;
2. as a vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$; or
3. as a matrix equation $A\mathbf{x} = \mathbf{b}$.

THEOREM 3

If A is a $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbf{R}^m$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}.$$

Useful Fact:

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a

____________ ____________________ of the columns of A.

EXAMPLE: Let $A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all $\mathbf{b}$? No, only if $-2b_1 + b_3 = 0$.

Solution: Augmented matrix corresponding to $A\mathbf{x} = \mathbf{b}$:

$$\left[\begin{array}{ccc|c} 1 & 4 & 5 & b_1 \\ -3 & -11 & -14 & b_2 \\ 2 & 8 & 10 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} \textcircled{1} & 4 & 5 & b_1 \\ 0 & \textcircled{1} & 1 & 3b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 + b_3 \end{array}\right]$$

augmented matrix   echelon form

$A\mathbf{x} = \mathbf{b}$ is ________ consistent for all $\mathbf{b}$ since some choices of $\mathbf{b}$ make $-2b_1 + b_3$ nonzero.

$A\underline{x} = \underline{b}$ is a linear system, with augmented matrix

$$(A \,\vdots\, \underline{b}) = \left(\begin{array}{ccc|c} 1 & 4 & 5 & b_1 \\ -3 & -11 & -14 & b_2 \\ 2 & 8 & 10 & b_3 \end{array}\right)$$

$-2b_1 + b_3 = 0$: $\left(\begin{array}{ccc|c} \textcircled{1} & 4 & 5 & b_1 \\ 0 & \textcircled{1} & 1 & 3b_1 + b_2 \\ 0 & 0 & 0 & 0 \end{array}\right)$ infinitely many solutions

$-2b_1 + b_3 \neq 0$: $\left(\begin{array}{ccc|c} \textcircled{1} & 4 & 5 & b_1 \\ 0 & \textcircled{1} & 1 & 3b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 + b_3 \end{array}\right)$ no solutions

$$A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{bmatrix}$$

$$\uparrow \quad \uparrow \quad \uparrow$$

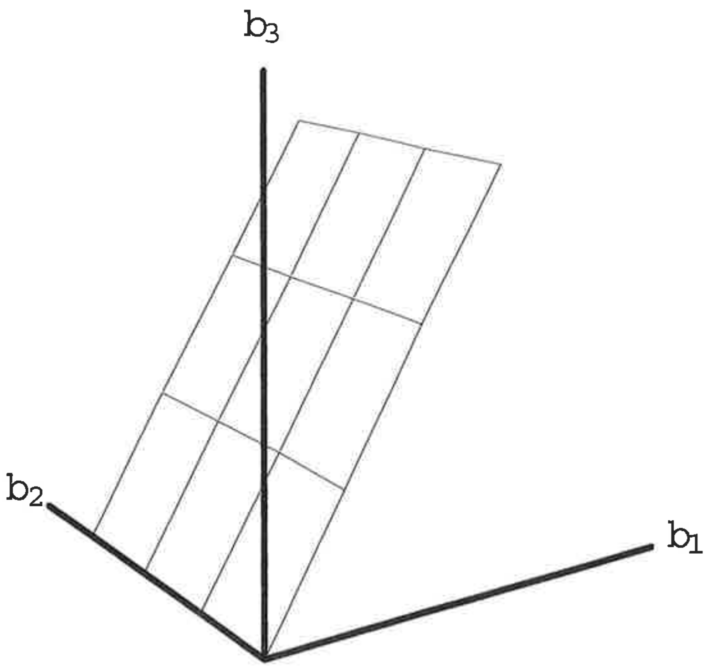
$$\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3$$

The equation $A\mathbf{x} = \mathbf{b}$ is consistent if

$$-2b_1 + b_3 = 0.$$

(equation of a plane in $\mathbf{R}^3$)

$x_1\mathbf{a}_1 + x_2\mathbf{a}_3 + x_3\mathbf{a}_3 = \mathbf{b}$ if and only if $b_3 - 2b_1 = 0$.



Columns of A span a plane
in $\mathbf{R}^3$ through $\mathbf{0}$

Instead, if *any* **b** in $\mathbf{R}^3$ (not just those lying on a particular line or in a plane) can be expressed as a linear combination of the columns of A, then we say that the columns of A span $\mathbf{R}^3$.

Definition

We say that **the columns of** $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_p \end{bmatrix}$ **span** $\mathbf{R}^m$ if ***every*** vector **b** in $\mathbf{R}^m$ is a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_p$

(i.e. Span$\{\mathbf{a}_1, \ldots, \mathbf{a}_p\} = \mathbf{R}^m$).

THEOREM 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent:

a. For each **b** in $\mathbf{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each **b** in $\mathbf{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbf{R}^m$.

d. A has a pivot position in every row.

Proof (outline): Statements (a), (b) and (c) are logically equivalent.

To complete the proof, we need to show that (a) is true when (d) is true and (a) is false when (d) is false.

Suppose (d) is _______________. Then row-reduce the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$:

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} \sim \cdots \sim \begin{bmatrix} U & \mathbf{d} \end{bmatrix}$$

and each row of U has a pivot position and so there is no pivot in the last column of $\begin{bmatrix} U & \mathbf{d} \end{bmatrix}$.

So (a) is _______________.

Now suppose (d) is _______________. Then the last row of $\begin{bmatrix} U & \mathbf{d} \end{bmatrix}$ contains all zeros.

Suppose **d** is a vector with a 1 as the last entry. Then $\begin{bmatrix} U & \mathbf{d} \end{bmatrix}$ represents an inconsistent system.

Row operations are reversible: $\begin{bmatrix} U & \mathbf{d} \end{bmatrix} \sim \cdots \sim \begin{bmatrix} A & \mathbf{b} \end{bmatrix}$

$\Rightarrow \begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ is inconsistent also. So (a) is _______________. ■

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible $\mathbf{b}$?

Solution: A has only _____ columns and therefore has at most _____ pivots.

Since A does not have a pivot in every ________, $A\mathbf{x} = \mathbf{b}$ is __________ ________________________ for all possible $\mathbf{b}$,

according to Theorem 4.

EXAMPLE: Do the columns of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 3 & 9 \end{bmatrix}$ span $\mathbf{R}^3$?

Solution:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 3 & 9 \end{bmatrix} \sim$$

(no pivot in row 2)

By Theorem 4, the columns of A

____.

Another method for computing $A\mathbf{x}$

Read Example 4 on page 44 through Example 5 on page 45 to learn this rule for computing the product $A\mathbf{x}$.

Theorem 5

If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbf{R}^n$, and c is a scalar, then:

a. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

b. $A(c\mathbf{u}) = cA\mathbf{u}$.

2.1 Matrix Operations

Matrix Notation:

Two ways to denote $m \times n$ matrix A:

In terms of the *columns* of A:

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$$

In terms of the *entries* of A:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Main diagonal entries: $a_{11}\ a_{22}\ a_{33} \cdots a_{nn}$

Zero matrix:

$$0 = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

THEOREM 1

Let A, B, and C be matrices of the same size, and let r and s be scalars. Then

a. $A + B = B + A$

b. $(A + B) + C = A + (B + C)$

c. $A + 0 = A$

d. $r(A + B) = rA + rB$

e. $(r + s)A = rA + sA$

f. $r(sA) = (rs)A$

Matrix Multiplication

Multiplying B and $\mathbf{x}$ transforms $\mathbf{x}$ into the vector $B\mathbf{x}$. In turn, if we multiply A and $B\mathbf{x}$, we transform $B\mathbf{x}$ into $A(B\mathbf{x})$. So $A(B\mathbf{x})$ is the composition of two mappings.

Define the product AB so that $A(B\mathbf{x}) = (AB)\mathbf{x}$.

Suppose A is $m \times n$ and B is $n \times p$ where

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}.$$

Then

$$B\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_p\mathbf{b}_p$$

and

$$A(B\mathbf{x}) = A(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_p\mathbf{b}_p)$$

$$= A(x_1\mathbf{b}_1) + A(x_2\mathbf{b}_2) + \cdots + A(x_p\mathbf{b}_p)$$

$$= x_1A\mathbf{b}_1 + x_2A\mathbf{b}_2 + \cdots + x_pA\mathbf{b}_p = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}.$$

Therefore,

$$A(B\mathbf{x}) = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}\mathbf{x}.$$

and by defining

$$\boxed{AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}}$$

we have $A(B\mathbf{x}) = (AB)\mathbf{x}$.

EXAMPLE: Compute AB where $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix}$.

Solution:

$$A\mathbf{b}_1 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \qquad A\mathbf{b}_2 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ -24 \\ 6 \end{bmatrix} \qquad\qquad = \begin{bmatrix} 2 \\ 26 \\ -7 \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} -4 & 2 \\ -24 & 26 \\ 6 & -7 \end{bmatrix}$$

Note that $A\mathbf{b}_1$ is a linear combination of the columns of A and $A\mathbf{b}_2$ is a linear combination of the columns of A.

> Each column of AB is a linear combination of the columns of A using weights from the corresponding columns of B.

EXAMPLE: If A is 4×3 and B is 3×2, then what are the sizes of AB and BA?

Solution:

$$AB = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} = \begin{bmatrix} \quad & \quad \end{bmatrix}$$

BA would be $\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$

which is ______________________.

If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$.

Row-Column Rule for Computing AB (alternate method)

The definition

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}$$

is good for theoretical work.

When A and B have small sizes, the following method is more efficient when working by hand.

If AB is defined, let $(AB)_{ij}$ denote the entry in the ith row and jth column of AB. Then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

$$\begin{bmatrix} & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{bmatrix} \begin{bmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \\ & b_{nj} & \end{bmatrix}$$

$$= \begin{bmatrix} & & \\ & (AB)_{ij} & \\ & & \end{bmatrix}$$

EXAMPLE $A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix}$. Compute AB, if it is defined.

Solution: Since A is 2×3 and B is 3×2, then AB is defined and AB is ______×______.

$$AB = \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{6} \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{2} & -3 \\ \mathbf{0} & 1 \\ \mathbf{4} & -7 \end{bmatrix} = \begin{bmatrix} \mathbf{28} & \blacksquare \\ \blacksquare & \blacksquare \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{6} \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & \mathbf{-3} \\ 0 & \mathbf{1} \\ 4 & \mathbf{-7} \end{bmatrix} = \begin{bmatrix} 28 & \mathbf{-45} \\ \blacksquare & \blacksquare \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 6 \\ \mathbf{-1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{2} & -3 \\ \mathbf{0} & 1 \\ \mathbf{4} & -7 \end{bmatrix} = \begin{bmatrix} 28 & -45 \\ \mathbf{2} & \blacksquare \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 6 \\ \mathbf{-1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} 2 & \mathbf{-3} \\ 0 & \mathbf{1} \\ 4 & \mathbf{-7} \end{bmatrix} = \begin{bmatrix} 28 & -45 \\ 2 & \mathbf{-4} \end{bmatrix}$$

So $AB = \begin{bmatrix} 28 & -45 \\ 2 & -4 \end{bmatrix}$.

THEOREM 2

Let A be $m \times n$ and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B + C) = AB + AC$ (left - distributive law)

c. $(B + C)A = BA + CA$ (right-distributive law)

d. $r(AB) = (rA)B = A(rB)$

for any scalar r

e. $I_m A = A = AI_n$ (identity for matrix multiplication)

WARNINGS

Properties above are analogous to properties of real numbers. But **NOT ALL** real number properties correspond to matrix properties.

1. It is not the case that AB always equal BA. (see Example 7, page 114)
2. Even if $AB = AC$, then B may not equal C. (see Exercise 10, page 116)
3. It is possible for $AB = 0$ even if $A \neq 0$ and $B \neq 0$. (see Exercise 12, page 116)

Powers of A

$$A^k = \underbrace{A \cdots A}_{k}$$

EXAMPLE:

$$\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 21 & 8 \end{bmatrix}$$

If A is $m \times n$, the **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 8 \\ 7 & 6 & 5 & 4 & 3 \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} 1 & 6 & 7 \\ 2 & 7 & 6 \\ 3 & 8 & 5 \\ 4 & 9 & 4 \\ 5 & 8 & 3 \end{bmatrix}$$

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}$. Compute AB, $(AB)^T$, A^TB^T and B^TA^T.

Solution:

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

$$A^TB^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 10 \\ 2 & 0 & -4 \\ 2 & 1 & 4 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

THEOREM 3

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$ (I.e., the transpose of A^T is A)

b. $(A+B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$ (I.e. the transpose of a product of matrices equals the product of their transposes in reverse order.)

EXAMPLE: Prove that $(ABC)^T =$ _________.

Solution: By Theorem 3d,

$$(ABC)^T = ((AB)C)^T = C^T(\qquad)^T$$

$$= C^T(\qquad) = \_\_\_\_\_\_\_\_.$$

(3) Determinants:

A (n×n matrix) $\rightsquigarrow$ det(A) = |A| (a number)

Ex: n=2

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow$ $\boxed{\det A = ad - bc}$

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \rightsquigarrow \det A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1\cdot 4 - 2\cdot 3$
$= 4 - 6 = \underline{\underline{-2}}$

Ex: n=3

$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$

Method for computing det(A) using cofactors:

Cofactor expansion along the first row:

$$\boxed{\det(A) = 1 \cdot C_{11} + 1 \cdot C_{12} + 1 \cdot C_{13}}$$

Defn. of C_{ij}: $C_{ij} = (-1)^{i+j} \cdot M_{ij}$

M_{ij}: Minor at position (i,j)
= determinant of the submatrix of A, where row i and col j is deleted

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} : \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \quad \begin{pmatrix} * & * & * \\ 1 & 2 & * \\ 1 & 3 & * \end{pmatrix}$$

Sign of cofactors:

$$(-1)^{i+j} \qquad \begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \\ + & - & + & - & \\ \vdots & & & & \end{pmatrix}$$

$$|A| = 1\cdot(+1)\cdot\begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} + 1\cdot(-1)\cdot\begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 1\cdot(+1)\cdot\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 1\cdot(2\cdot 9 - 3\cdot 4) - (1\cdot 9 - 1\cdot 4) + (1\cdot 3 - 1\cdot 2)$$

$$= 6 - 5 + 1 = \underline{\underline{2}}$$

Fact:

You can find any determinant by <u>cofactor expansion</u> using any row or any column.

Problems:

1) Is $\begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix}$ a linear combinatión of the vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$?

2) Compute the determinant

$$\begin{vmatrix} 1 & 3 & 7 \\ 2 & 4 & 0 \\ 1 & 3 & 1 \end{vmatrix}$$

3) Compute the determinant

$$\begin{vmatrix} 1 & 4 & 0 & 6 \\ 7 & -1 & 0 & -1 \\ 1 & 0 & 3 & 4 \\ 2 & 4 & 1 & 0 \end{vmatrix}$$

4) Compute

$$\begin{pmatrix} 1 & 2 & 4 \\ 3 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 7 & 4 \end{pmatrix}$$