LECTURE 1

FORK 1003

LINEAR ALGEBRA

EIVIND ERIKSEN 7TH AUG 2013

SUMMARY:

- LINEAR SYSTEMS (OF EQUATIONS)
- SOLVING LINEAR SYSTEMS BY GAUSSIAN ELIMATION
- ~~MATRICES, MATRIX ALGEBRA, VECTORS~~

Relevant part of textbook:

[ME] 6.1–6.2, 7.1–7.3

1 LINEAR SYSTEMS

Ex:
$$\begin{cases} x+y+z=4 \\ x-y+5z=1 \\ 2x+y-z=3 \end{cases}$$
(3×3)

$$\begin{cases} x+y-z-w=1 \\ x+z=4 \end{cases}$$
(2×4)

In general:

$$\begin{cases} a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n=b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1+a_{m2}x_2+\dots+a_{mn}x_n=b_m \end{cases}$$

$(m\times n)$ m equations
n unknowns $(x_1, x_2, \dots, x_n)$

(2) Gaussian elimination

- A method for solving linear systems
- efficient
- good for pedagogical reasons

Ex: $\begin{cases} x_1 + x_2 = 3 & \text{i)} \\ x_1 - x_2 = 1 & \text{ii)} \end{cases}$

Substitution:

i) $x_2 = 3 - x_1$ (solve for x_2)

ii) $x_1 - (3 - x_1) = 1$ (substitute for x_2)

$2x_1 - 3 = 1$

$2x_1 = 4$

$x_1 = 2$ $\quad x_2 = 3 - 2 = 1$

Elimination

$$\left.\begin{array}{ll} (1) & x_1 + x_2 = 3 \\ (2) & x_1 - x_2 = 1 \end{array}\right\}$$

$$\downarrow$$

$$\left.\begin{array}{ll} (1)+(2) & 2x_1 \quad\quad = 4 \\ (2) & x_1 - x_2 = 1 \end{array}\right\} \quad \begin{array}{l} x_1 = 2 \\ x_2 = 1 \end{array}$$

Section 1.1: Systems of Linear Equations

A **linear equation:**

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

EXAMPLE:

$$4x_1 - 5x_2 + 2 = x_1 \quad \text{and} \quad x_2 = 2(\sqrt{6} - x_1) + x_3$$

$\downarrow$ rearranged $\downarrow$ $\downarrow$ rearranged $\downarrow$

$$3x_1 - 5x_2 = -2 \qquad\qquad 2x_1 + x_2 - x_3 = 2\sqrt{6}$$

Not linear:

$$4x_1 - 6x_2 = x_1x_2 \quad \text{and} \quad x_2 = 2\sqrt{x_1} - 7$$

A **system of linear equations** (or a **linear system**):

A collection of one or more linear equations involving the same set of variables, say, $x_1, x_2, \ldots, x_n$.

A **solution** of a linear system:

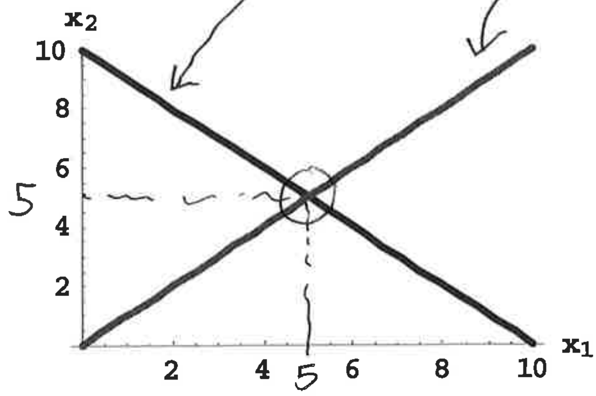
A list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation in the system true when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively.

EXAMPLE Two equations in two variables:

$$\begin{array}{rcrcr} x_1 & + & x_2 & = & 10 \\ -x_1 & + & x_2 & = & 0 \end{array}$$

$x_2 = 10 - x_1$

$\boxed{x_2 = x_1}$

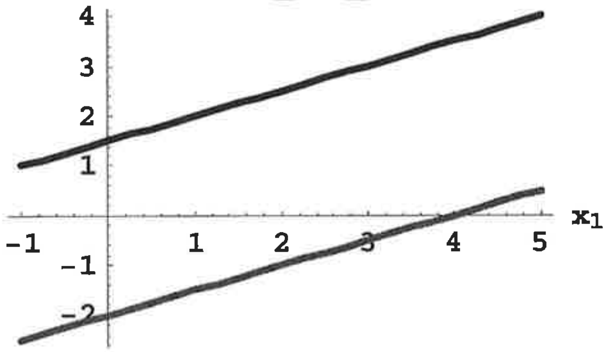


one unique solution

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -3 \\ 2x_1 & - & 4x_2 & = & 8 \end{array}$$

$\cdot 2$

$2x_1 - 4x_2 = -6$

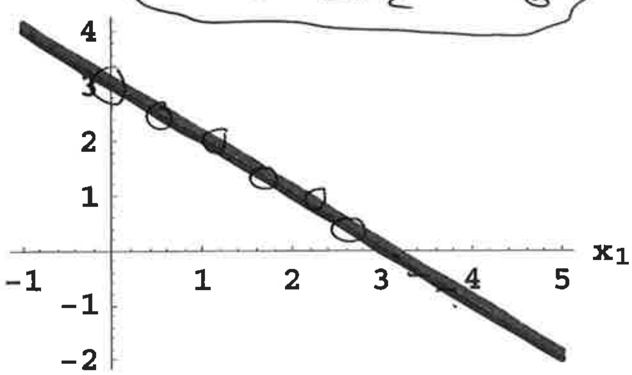


no solution

$$\begin{array}{rcrcr} x_1 & + & x_2 & = & 3 \\ -2x_1 & - & 2x_2 & = & -6 \end{array}$$

(-2)

$-2x_1 - 2x_2 = -6$



infinitely many solutions

two soln's
<u>not</u> linear

BASIC FACT: A system of linear equations has either

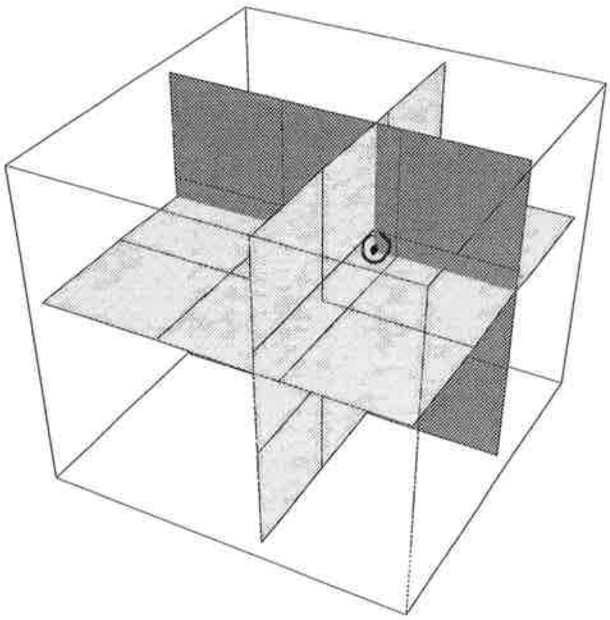
(i) exactly one solution *(consistent)* or

(ii) infinitely many solutions *(consistent)* or

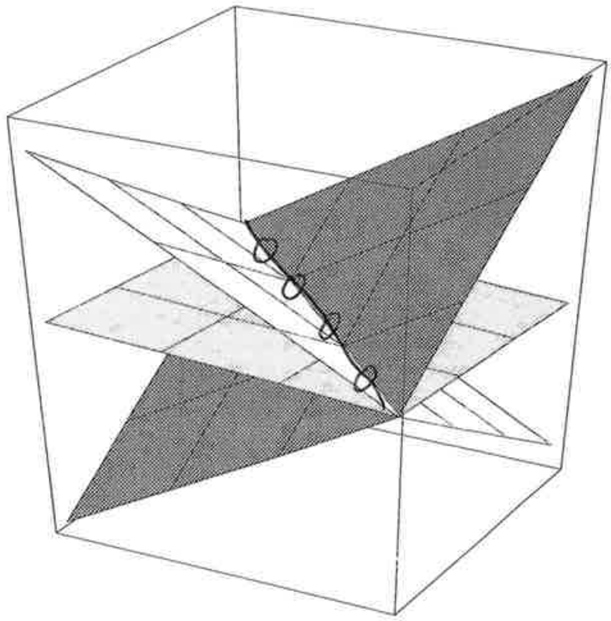
(iii) no solution *(inconsistent)*.

EXAMPLE: Three equations in three variables. Each equation determines a plane in 3-space.

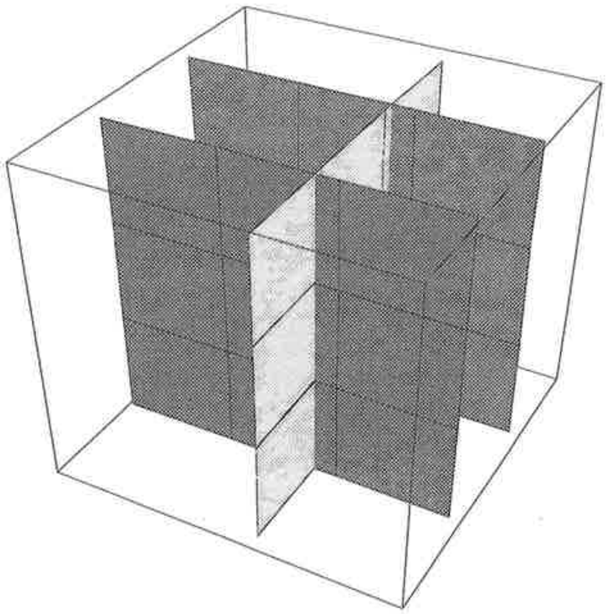
i) The planes intersect in one point. *(one solution)*



ii) The planes intersect in one line. *(infinitely many solutions)*



iii) There is not point in common to all three planes. *(no solution)*



The **solution set:**

- The set of all possible solutions of a linear system.

Equivalent systems:

- Two linear systems with the same solution set.

STRATEGY FOR SOLVING A SYSTEM:

- *Replace one system with an equivalent system that is easier to solve.*

EXAMPLE:

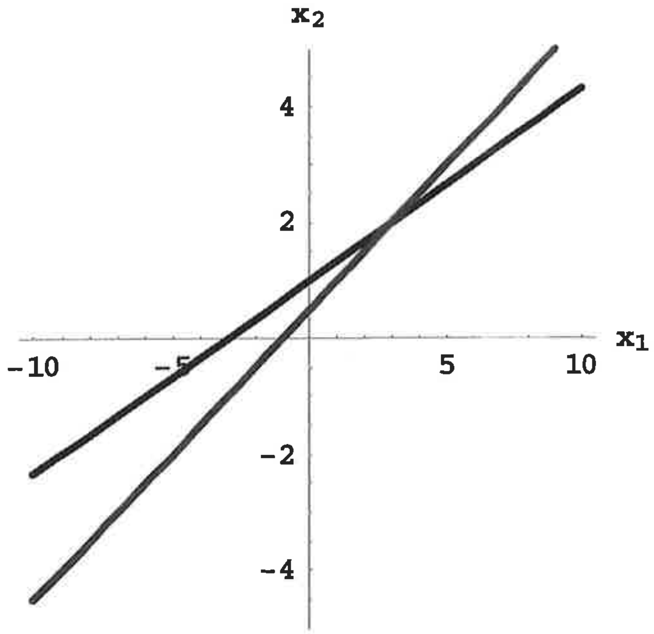
$$\left.\begin{array}{lrcrcr} (1) & x_1 & - & 2x_2 & = & -1 \\ (2) & -x_1 & + & 3x_2 & = & 3 \end{array}\right\}$$

$$\downarrow$$

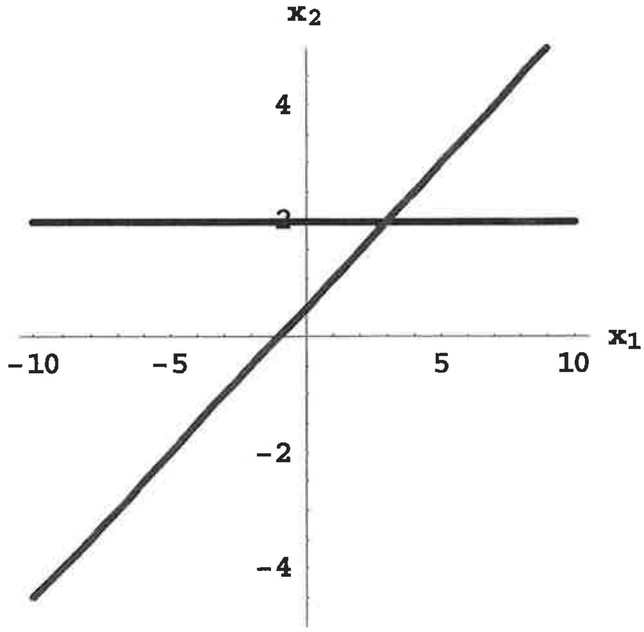
$$\left.\begin{array}{llrcrcr} (a) & (1) & x_1 & - & 2x_2 & = & -1 \\ (b) & (1)+(2) & & & x_2 & = & 2 \end{array}\right\} \qquad \begin{array}{l} x_1 = 3 \\ x_2 = 2 \end{array}$$

$$\downarrow$$

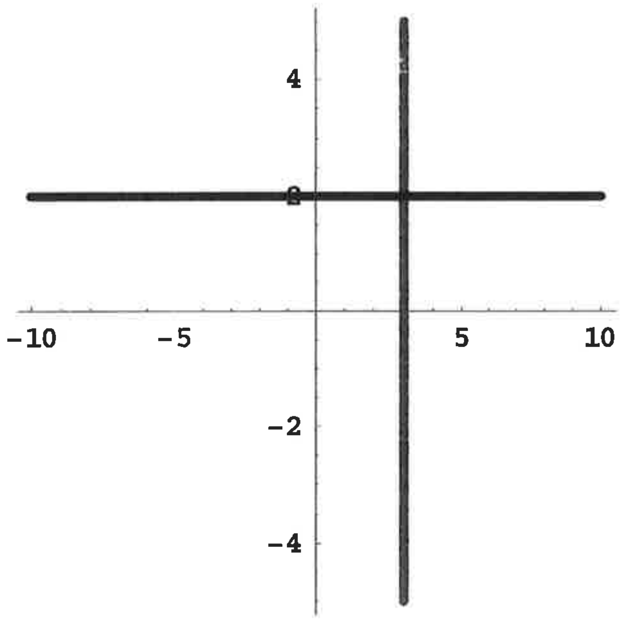
$$\left.\begin{array}{lrcr} (a)+2(b) & x_1 & & = 3 \\ (b) & & x_2 & = 2 \end{array}\right\} \qquad \begin{array}{l} x_1 = 3 \\ x_2 = 2 \end{array}$$



$$\left.\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array}\right\}$$



$$\left.\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ & & x_2 & = & 2 \end{array}\right\}$$



$$\left.\begin{array}{rcrcr} x_1 & & & = & 3 \\ & & x_2 & = & 2 \end{array}\right\}$$

Matrix Notation

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

(coefficient matrix)

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \qquad \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & \\ \downarrow & \downarrow & \end{array} \\ \left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right] \end{array}$$

(augmented matrix)

$$\downarrow \qquad\qquad \downarrow$$

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ x_2 &= 2 \end{aligned} \qquad \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 2 \end{array}\right]$$

$$\downarrow \qquad\qquad \downarrow$$

$$\begin{aligned} x_1 \quad &= 3 \\ x_2 &= 2 \end{aligned} \qquad \begin{array}{c} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right] \\ \begin{array}{ccc} \uparrow & \uparrow & \\ x_1 & x_2 & \end{array} \end{array} \qquad \begin{aligned} x_1 &= 3 \\ x_2 &= 2 \end{aligned}$$

Facts: $\begin{cases} \text{- Produce equiv. lin. system} \\ \text{- All lin. systems can be solved using row operations.} \end{cases}$

Elementary Row Operations:
1. *(Replacement)* Add one row to a multiple of another row.
2. *(Interchange)* Interchange two rows.
3. *(Scaling)* Multiply all entries in a row by a nonzero constant.

Row equivalent matrices: Two matrices where one matrix can be transformed into the other matrix by a sequence of elementary row operations.

Fact about Row Equivalence: If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

$$\left.\begin{array}{l} x_1 + x_2 = 3 \\ x_1 - x_2 = 1 \end{array}\right\} \longleftrightarrow \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 1 \end{array}\right)$$

lin. system

augmented matrix

Ex:

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \quad\quad 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases} \longleftrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right)$$

Row operations:

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right) \cdot 1/2 \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ -4 & 5 & 9 & -9 \end{array}\right)$$ scaling

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \\ 0 & 2 & -8 & 8 \end{array}\right)$$ inter-change

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right) \cdot 4 \longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right)$$ replace-ment

EXAMPLE:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \; 4$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \; \cdot \tfrac{1}{2}$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right] \; 3$$

use back-substitution

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$x_1 = 29$

$x_2 = 16$

$x_3 = 3$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & & & = & -3 \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$
\begin{array}{rcl}
x_1 & & = 29 \\
& x_2 & = 16 \\
& & x_3 = 3
\end{array}
\qquad
\left[\begin{array}{ccc|c}
1 & 0 & 0 & 29 \\
0 & 1 & 0 & 16 \\
0 & 0 & 1 & 3
\end{array}\right]
$$

Solution: $(29, 16, 3)$

Check: Is $(29, 16, 3)$ a solution of the *original* system?

$$
\begin{array}{rcrcrcr}
x_1 & - & 2x_2 & + & x_3 & = & 0 \\
 & & 2x_2 & - & 8x_3 & = & 8 \\
-4x_1 & + & 5x_2 & + & 9x_3 & = & -9
\end{array}
$$

$$
\begin{array}{rcll}
(29) - 2(16) + 3 & = 29 - 32 + 3 & = & 0 \\
2(16) - 8(3) & = 32 - 24 & = & 8 \\
-4(29) + 5(16) + 9(3) & = -116 + 80 + 27 & = & -9
\end{array}
$$

Gaussian elimination:

row operation + back substitution

Gauss-Jordan elimination: (variation)

row operations + more row operations

Two Fundamental Questions (Existence and Uniqueness)

1) Is the system consistent; (i.e. does a solution **exist**?)

2) If a solution exists, is it **unique**? (i.e. is there one & only one solution?)

EXAMPLE: Is this system consistent?

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

In the last example, this system was reduced to the triangular form:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

This is sufficient to see that the system is consistent and unique. Why?

EXAMPLE: Is this system consistent?

$$\begin{aligned} 3x_2 - 6x_3 &= 8 \\ x_1 - 2x_2 + 3x_3 &= -1 \\ 5x_1 - 7x_2 + 9x_3 &= 0 \end{aligned} \qquad \begin{bmatrix} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{bmatrix}$$

Solution: Row operations produce:

$$\begin{bmatrix} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 3 & -6 & 5 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

Equation notation of triangular form:

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -1 \\ 3x_2 - 6x_3 &= 8 \\ 0x_3 &= -3 \quad \leftarrow \textit{Never true} \end{aligned}$$

The original system is inconsistent!

EXAMPLE: For what values of h will the following system be consistent?

$$\begin{array}{rcrcl} 3x_1 & - & 9x_2 & = & 4 \\ -2x_1 & + & 6x_2 & = & h \end{array}$$

Solution: Reduce to triangular form.

$$\left[\begin{array}{rr|r} 3 & -9 & 4 \\ -2 & 6 & h \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -3 & \frac{4}{3} \\ -2 & 6 & h \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -3 & \frac{4}{3} \\ 0 & 0 & h+\frac{8}{3} \end{array}\right]$$

The second equation is $0x_1 + 0x_2 = h + \frac{8}{3}$. System is consistent only if $h + \frac{8}{3} = 0$ or $h = \frac{-8}{3}$.

Ex: $\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right)$ not echelon form

Section 1.2: Row Reduction and Echelon Forms

Echelon form (or row echelon form):

1. All nonzero rows are above any rows of all zeros.
2. Each *leading entry* (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zero.

EXAMPLE: Echelon forms

(a) $\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ echelon form

(b) $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$ echelon form

(c) $\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \end{bmatrix}$ echelon form

Reduced echelon form: Add the following conditions to conditions 1, 2, and 3 above:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

reduced echelon form is unique

EXAMPLE (continued):

Reduced echelon form :

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 1 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$ reduced echelon form

Theorem 1 (Uniqueness of The Reduced Echelon Form):

Each matrix is row-equivalent to one and only one reduced echelon matrix.

Important Terms:

- **pivot position:** a position of a leading entry in an echelon form of the matrix.

- **pivot:** a nonzero number that either is used in a pivot position to create 0's or is changed into a leading 1, which in turn is used to create 0's. = leading entry in an echelon form

- **pivot column:** a column that contains a pivot position.

(See the Glossary at the back of the textbook.)

pivots

$$-3 \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 3 & 4 & 7 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 4 & -5 \end{array}\right)$$

echelon form

EXAMPLE: Row reduce to echelon form and locate the pivot columns.

$$\left[\begin{array}{cccc|c} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{array}\right]$$

4x4

Solution

pivot

$$\left[\begin{array}{cccc|c} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{array}\right] \quad 1 \quad 2 \quad \left(\begin{array}{cccc|c} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{array}\right)$$

pivot column

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \quad -5 \quad 3 \quad \left(\begin{array}{cccc|c} 1 & 4 & 5 & -9 & - \\ 0 & 1 & 2 & -3 & - \\ 0 & 5 & 10 & -15 & - \\ 0 & -3 & -6 & 4 & \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \textbf{Possible Pivots:} \quad \longleftarrow \quad \left(\begin{array}{cccc|c} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{array}\right) 4$$

echelon form

pivot positions

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \sim \begin{array}{c} \left[\begin{array}{cccc|c} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ x_1 & x_2 & x_3 & x_4 \end{array} \end{array}$$

Original Matrix:

$$\left[\begin{array}{cccc|c} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{array}\right]$$

$$\begin{array}{cccc} \uparrow & \uparrow & & \uparrow \\ 1 & 2 & & 4 \\ x_1 & x_2 & & x_4 \end{array}$$

pivot columns: 1 2 4

Note: There is no more than one pivot in any row. There is no more than one pivot in any column.

Fact: no solutions $\Longleftrightarrow$ the right column is a pivot column

consistent $\Longleftrightarrow$ the right column is not a pivot column

Ex:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

$$x_1 + 2x_2 + 3x_3 = 4$$
$$x_2 + 2x_3 = 7$$
$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 3$$

no solutions

If there are solutions (consistent):

one solution $\Longleftrightarrow$ every column (but the last) has a pivot

infinitely many solutions $\Longleftrightarrow$ not every column (but the last) has a pivot

Ex:

$$\left(\begin{array}{cccc|c} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \uparrow & \uparrow & & \uparrow & \end{array}\right)$$

infinitely many solutions (col. 3 lacks a pivot)

$$\begin{aligned} x_1 + 4x_2 + 5x_3 - 9x_4 &= -7 \\ 2x_2 + 4x_3 - 6x_4 &= -6 \\ -5x_4 &= 0 \end{aligned}$$

x_1, x_2, x_4 : basic var.
x_3 : free var.

$$\boxed{\begin{aligned} x_1 &= 3x_3 + 5 \\ x_2 &= -2x_3 - 3 \\ x_3 &= \text{free} \\ x_4 &= 0 \end{aligned}}$$

$$\left\{\begin{aligned} x_1 &= -4(-2x_3 - 3) \overset{-7}{\div} 5x_3 \\ x_2 &= -2x_3 - 3 \\ x_4 &= 0 \end{aligned}\right.$$

EXAMPLE: Row reduce to echelon form and then to reduced echelon form:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Cover the top row and look at the remaining two rows for the left-most nonzero column.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \text{ (echelon form)}$$

Final step to create the reduced echelon form:
Beginning with the rightmost leading entry, and working upwards to the left, create zeros above each leading entry and scale rows to transform each leading entry into 1.

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

SOLUTIONS OF LINEAR SYSTEMS

- **basic variable:** any variable that corresponds to a pivot column in the augmented matrix of a system.
- **free variable:** all nonbasic variables.

EXAMPLE:

$$\left[\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right] \qquad \begin{aligned} x_1 + 6x_2 \qquad\qquad + 3x_4 \qquad &= 0 \\ x_3 - 8x_4 \qquad &= 5 \\ x_5 &= 7 \end{aligned}$$

pivot columns: 1, 3, 5

basic variables: x_1, x_3, x_5

free variables: x_2, x_4

Final Step in Solving a Consistent Linear System: After the augmented matrix is in **reduced** echelon form and the system is written down as a set of equations:

Solve each equation for the basic variable in terms of the free variables (if any) in the equation.

EXAMPLE:

$$\begin{array}{rrrrrl} \underline{x_1} & +6x_2 & & +3x_4 & & = 0 \\ & & \underline{x_3} & -8x_4 & & = 5 \\ & & & & \underline{x_5} & = 7 \end{array} \qquad \left\{ \begin{array}{l} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 8x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{array} \right.$$

(general solution)

The **general solution** of the system provides a parametric description of the solution set. (The free variables act as parameters.) The above system has **infinitely many solutions.**

Why?

Warning: Use only the reduced echelon form to solve a system.

(with echelon form, it is also possible but it requires some work in the back substitution)

Existence and Uniqueness Questions

EXAMPLE:

$$\left[\begin{array}{rrrrrl} & 3x_2 & -6x_3 & +6x_4 & +4x_5 & = -5 \\ 3x_1 & -7x_2 & +8x_3 & -5x_4 & +8x_5 & = 9 \\ 3x_1 & -9x_2 & +12x_3 & -9x_4 & +6x_5 & = 15 \end{array}\right]$$

In an earlier example, we obtained the echelon form:

$$\left[\begin{array}{rrrrr|r} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \quad (x_5 = 4)$$

No equation of the form $0 = c$, where $c \neq 0$, so the system is consistent.

Free variables: x_3 and x_4

Consistent system with free variables	$\Rightarrow$ **infinitely many solutions.**

EXAMPLE:

$$\begin{array}{rrcr} 3x_1 & +4x_2 & = & -3 \\ 2x_1 & +5x_2 & = & 5 \\ -2x_1 & -3x_2 & = & 1 \end{array} \rightarrow \begin{bmatrix} 3 & 4 & -3 \\ 2 & 5 & 5 \\ -2 & -3 & 1 \end{bmatrix}$$

$$\sim \left[\begin{array}{cc|c} 3 & 4 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} 3x_1 + 4x_2 = -3 \\ x_2 = 3 \end{array}$$

Consistent system, no free variables

⇒ unique solution.

Theorem 2 (Existence and Uniqueness Theorem)

1. A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column, i.e., if and only if an echelon form of the augmented matrix has no row of the form

$$\begin{bmatrix} 0 & \dots & 0 & b \end{bmatrix} \text{ (where } b \text{ is nonzero).}$$

2. If a linear system is consistent, then the solution contains either
(i) a unique solution (when there are no free variables) or
(ii) infinitely many solutions (when there is at least one free variable).

Using Row Reduction to Solve Linear Systems

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If not, stop; otherwise go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. State the solution by expressing each basic variable in terms of the free variables and declare the free variables.

EXAMPLE:

a) What is the largest possible number of pivots a 4×6 matrix can have? Why?

4

b) What is the largest possible number of pivots a 6×4 matrix can have? Why?

4

c) How many solutions does a consistent linear system of 3 equations and 4 unknowns have? Why?

$$\left(\begin{array}{cccc|c} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right)$$

infinitely many
(at least one free var.)

d) Suppose the coefficient matrix corresponding to a linear system is 4×6 and has 3 pivot columns. How many pivot columns does the augmented matrix have if the linear system is inconsistent?

$$\left(\begin{array}{cccccc|c} \odot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \odot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \odot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{array}\right)$$

4

Problems: Solve the linear systems using Gaussian elimination:

1. $$\begin{aligned} x + y + z &= 3 \\ x + 2y + 4z &= 7 \\ x + 3y + 9z &= 13 \end{aligned}$$

2. $$\begin{aligned} x + y + z &= 4 \\ x - y + 2z &= 1 \\ x + 5y - z &= 10 \end{aligned}$$

3. $$\begin{aligned} x + y + z + w &= 4 \\ x - y + z - w &= 1 \\ x + 3y + z + 3w &= 7 \end{aligned}$$