# SYSTEMS OF LINEAR EQUATIONS ANSWERS AND COMMENTS TO SELECTED EXERCISES

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## 1. First lecture

**Exercise 1.1.** Which of these equations are linear equations?

$$\text{(c)} \quad \lambda x - \beta y - z = t$$

Well, here the answer depends on which of the letters denotes *variables* and which denotes *parametres*. In mathematical models one very often have parametres which are 'undetermined but fixed' numbers which in principal may be put to any specific value. The solutions of equations with parametres will give formulas for the variables which contain the parametres. For instance if $x$, $y$ and $z$ are variables and $\lambda$ and $\beta$ are parametres (greek letters often denote parametres) then (c) is a linear equation regardless of whether $t$ is a variable or a parametre. But if both $\lambda$ and $x$ (or both $\beta$ and $y$) are variables then (c) is not a linear equation.

**Exercise 1.2.** Write up the augmented matrix of each of the following systems of linear equations.

$$\text{(c)} \begin{cases} 5x_2 + 9x_3 = -3 & \qquad x_1 + 22x_2 - 4x_3 = 5 \\ -4x_1 + 12x_2 - 4x_3 = -8 & \qquad -13x_1 + 5x_3 = -8 \end{cases}$$

The answer will depend on the ordering of the equations. If ordered from left to right, line by line we get

$$\begin{bmatrix} 0 & 5 & 9 & -3 \\ 1 & 22 & -4 & 5 \\ -4 & 12 & -4 & -8 \\ -13 & 0 & 5 & -8 \end{bmatrix}.$$

But if we ordered column by column then the two middle rows would be interchanged. Other orderings are also possible, giving different orders of the rows in the above matrix.

$$\text{(d)} \begin{cases} 15x_2 + 2x_3 = -3x_1 + x_4 - 7 \\ x_2 = -3x_3 + 1 \end{cases} \qquad \text{(e)} \begin{cases} 15x_2 + 2x_3 = -3x_2 + x_4 - 7 \\ x_2 = -3x_3 + 1 - x_1 \\ 0 = 2x_1 - 3x_3 + 11x_1 - 32 + 88x_3 \end{cases}$$

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In (d) and (e) the equations are not on standard form. An equation is on standard form if all variables are to the left of $=$, each variable only appearing once, and numbers (and parametres) not multiplied with variables to the right of $=$. The first equation in (d) is equivalent to $3x_1 + 15x_2 + 2x_3 - x_4 = -7$ which is on standard form. We get

$$(d): \quad \begin{bmatrix} 3 & 15 & 2 & -1 & -7 \\ 0 & 1 & 3 & 0 & 1 \end{bmatrix} \qquad (e): \quad \begin{bmatrix} 0 & 18 & 2 & -1 & -7 \\ 1 & 1 & 3 & 0 & 1 \\ 13 & 0 & 85 & 0 & 32 \end{bmatrix}$$

**Exercise 1.3.** Write up the linear systems corresponding to the following matrices.

$$(b) \quad \begin{bmatrix} 1 & 5 & 0 & 0 & 0 & 41 \\ 0 & 0 & 1 & -3 & 0 & 99 \\ 0 & 0 & 0 & 0 & 1 & 19 \end{bmatrix} \quad \text{gives} \quad \begin{cases} x_1 \; +5x_2 & = 41 \\ \qquad\quad x_3 \; -3x_4 & = 99 \\ \qquad\qquad\qquad x_5 & = 19 \end{cases}$$

**Exercise 1.4.** Note that a matrix on reduced echelon form is also on echelon form.
(a) Not echelon since 14 is a non-zero number above a pivot number.
(b) Reduced echelon. The pivot positions $(1,1)$, $(1,3)$ and $(3,5)$ all contain a 1 and the only zeros above them.
(c) Not echelon. In the second row the pivot position $(2,4)$ is to the right of the pivot position $(3,3)$ of the third row. (d) Reduced echelon. 1 in the only pivot position $(1,1)$, no further conditions.
(e) Echelon, but not reduced.
(f) Echelon, but not reduced.

**Exercise 1.5.** For each of the following matrices on reduced echelon form determine the pivot columns and the free variables.
(a) Pivot columns: 1 and 2, free variable $x_3$
(b) Pivot columns: 1, 3 and 4, free variables $x_2$ and $x_5$
(c) Pivot columns: 1, 2 and 3, no free variables
(d) Pivot columns: 1 and 3, free variable $x_2$ and $x_4$
(e) Pivot columns: 1, 2 and 4, free variable $x_3$
(f) Pivot columns: 1 and 3, free variables $x_2$ and $x_4$
Use this to solve the corresponding system of linear equations.

$$(d) \quad \begin{bmatrix} 1 & -25 & 0 & 3.5 & 400 \\ 0 & 0 & 1 & -40 & 300 \end{bmatrix} \quad \text{gives solution} \quad \begin{cases} x_2 \text{ free variable} \\ x_4 \text{ free variable} \\ x_3 = 40x_4 + 300 \\ x_1 = 25x_2 - 3.5x_4 + 400 \end{cases}$$

**Exercise 1.6.** Hege is doing elementary row operations. Explain in each case which operation she has performed and determine whether this brings the Gauss elimination forward or not.
(a) 5 times the first row added to the third. Positive step.
(c) $\frac{2}{3}$ of the first row is subtracted from the third. Positive step.

(d) Second and third row interchanged. Positive step. Further interchanging first and last row gives a matrix on echelon form.
(g) Third row added to first. Better to first eliminate the non-zero entries in the fifth column above pivot position $(4, 5)$.

...explain what Kåre does, determine whether the transformation is an elementary row operation, and determine whether the matrices are row equivalent.
(k) 3 (or perhaps $-3$) times the first row added to the third. This an elementary row operation, but the calculations are wrong. Not row equivalent. (By Gauss elimination on the matrices find the reduced echelon forms which are unique. They are different).
(l) 3 times the first row added to the second and $-2$ times the first row added to the third. This combines two elementary row operations, and the combination is not elementary. But the matrices are row equivalent since the steps could be performed after each other.
(m) The first row is subtracted from the third and at the same time the third is subtracted from the first. Each of these row operations is elementary, but this combination does not give a row equivalent matrix since the two operations cannot be performed as two elementary row operations, the one after the other (try!). Not row equivalent matrices since the reduced echelon form are different.
(n) The first column is added to the third. This is not a row operation. The matrices are not row equivalent (different reduced echelon forms).
(o) The first row is subtracted from the third and the first row has vanished (or multiplied by zero). Not an elementary row operation. Not row equivalent matrices (different reduced echelon forms).

**Exercise 1.7.** Use Gauss elimination to find row equivalent matrices on reduced echelon form.

$$\text{(a)} \begin{bmatrix} 1 & -2 & 0 & -2 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -2 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\text{(b)} \begin{bmatrix} 1 & -2 & 0 & -2 \\ 1 & -1 & -3 & -5 \\ -1 & 0 & 10 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -2 \\ 0 & 1 & -3 & -3 \\ -1 & 0 & 10 & 16 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & -2 \\ 0 & 1 & -3 & -3 \\ 0 & -2 & 10 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -2 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 4 & 8 \end{bmatrix} \text{ which equals (a).}$$

$$\text{(e)} \begin{bmatrix} 0 & 0 & 0 & 7 & 21 \\ 0 & 0 & 3 & -6 & -6 \\ 1 & 2 & -3 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 3 & 2 \\ 0 & 0 & 3 & -6 & -6 \\ 0 & 0 & 0 & 7 & 21 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 3 & 2 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 7 & 21 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 3 & 2 \\ 0 & 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 3 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 & -7 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\text{(f)} \begin{bmatrix} 4 & 8 & -6 & 7 & 17 \\ 1 & 2 & 0 & -3 & -4 \\ 3 & 6 & -9 & 9 & 6 \end{bmatrix} \sim \begin{bmatrix} 4 & 8 & -6 & 7 & 17 \\ 1 & 2 & 0 & -3 & -4 \\ 1 & 2 & -3 & 3 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 4 & 8 & -6 & 7 & 17 \\ 0 & 0 & 3 & -6 & -6 \\ 1 & 2 & -3 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 6 & -5 & 9 \\ 0 & 0 & 3 & -6 & -6 \\ 1 & 2 & -3 & 3 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 0 & 0 & 7 & 21 \\ 0 & 0 & 3 & -6 & -6 \\ 1 & 2 & -3 & 3 & 2 \end{bmatrix} \text{ which equals (e).}$$

**Exercise 1.8.** If we put $\gamma_1 = 93.53$, $\gamma_2 = 0.712$, $\gamma_3 = 95.05$ and $\gamma_5 = 34.30$, the equations (after giving them standard form) give us the following augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \gamma_1 \\ 0 & -\gamma_2 & 0 & 1 & \gamma_3 \\ \gamma_4 & 0 & -1 & \gamma_4 & \gamma_5 \\ 1 & -1 & -1 & 1 & 0 \end{bmatrix}$$

By some elementary row operations we obtain the following matrix on echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \gamma_1 \\ 0 & 1 & 1 & -1 & \gamma_1 \\ 0 & 0 & 1 & -\gamma_4 & \gamma_1\gamma_4 - \gamma_5 \\ 0 & 0 & 0 & 1 - \gamma_2 + \gamma_2\gamma_4 & \gamma_3 + \gamma_1\gamma_2 - \gamma_1\gamma_2\gamma_4 + \gamma_2\gamma_5 \end{bmatrix}$$

Solving 'backwards' beginning with the fourth row gives

$$\begin{aligned} c &= \frac{\gamma_3 + \gamma_1\gamma_2 - \gamma_1\gamma_2\gamma_4 + \gamma_2\gamma_5}{1 - \gamma_2 + \gamma_2\gamma_4} && = 438.3 \\ s &= \gamma_1\gamma_4 - \gamma_5 + \gamma_4 c && = 49.73 \\ y &= -s + c + \gamma_1 && = 482.1 \\ x &= \gamma_1 && = 93.53 \end{aligned}$$