LECTURE 2: FORK 1003

EIVIND ERIKSEN, AUG 9TH 2011 LINEAR ALGEBRA

SUMMARY:
- MATRIX MULTIPLICATION
- TRANSPOSE

} [EMEA] 15.2 – 15.5

- DETERMINANT

} [EMEA] 16.1 – 16.5

REVIEW: MATRIX–VECTOR multiplication

If $A = (\underline{a}_1 | \underline{a}_2 | \cdots | \underline{a}_n)$ is $m\times n$-matrix and $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is n-vector, then

$$\underset{m\times n}{A} \cdot \underline{x} = (\underline{a}_1 | \underline{a}_2 | \cdots | \underline{a}_n) \cdot \underset{n\times 1}{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}} := \underbrace{x_1 \cdot \underline{a}_1 + x_2 \cdot \underline{a}_2 + \cdots + x_n \cdot \underline{a}_n}_{m\times 1}$$

linear combination of columns of A with coeff. $x_1, \ldots, x_n$.

The lin. system $\underbrace{(A | \underline{b}) = (\underline{a}_1 | \underline{a}_2 | \cdots | \underline{a}_n | \underline{b})}_{\text{augmented matrix}}$ can be written $A \cdot \underline{x} = \underline{b}$.

Matrix multiplication:

If $A = (\underline{a}_1 | \underline{a}_2 | \cdots | \underline{a}_n)$ is $m\times n$-matrix, and $B = (\underline{b}_1 | \underline{b}_2 | \cdots | \underline{b}_p)$ is $n\times p$-matrix, then

$$\underset{m\times n}{A} \cdot \underset{n\times p}{B} = \underset{m\times p}{\underbrace{(A\cdot\underline{b}_1 | A\cdot\underline{b}_2 | \cdots | A\cdot\underline{b}_p)}_{\text{matrix–vector products}}}$$

Ex:

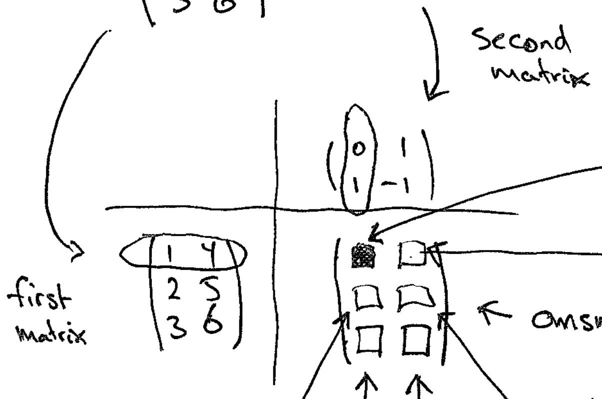
$$\underset{3\times 2}{\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}} \cdot \underset{2\times 2}{\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}} = \left(\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \;\middle|\; \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = \left(\begin{matrix} 4 \\ 5 \\ 6 \end{matrix} \;\middle|\; \begin{matrix} -3 \\ -3 \\ -3 \end{matrix} \right) = \underset{3\times 2}{\begin{pmatrix} 4 & -3 \\ 5 & -3 \\ 6 & -3 \end{pmatrix}}$$

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 1 \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

A better way of computing matrix mult:

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = ?$$



$$(1 \;\; 4) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \cdot 0 + 4 \cdot 1 = 0 + 4 = \underline{4}$$

$$(1 \;\; 4) \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \cdot 1 + 4 \cdot (-1) = 1 - 4 = \underline{-3}$$

$$(2 \;\; 5) \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2 \cdot 1 + 5 \cdot (-1) = 2 - 5 = \underline{-3}$$

$$(2 \;\; 5) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2 \cdot 0 + 5 \cdot 1 = 0 + 5 = \underline{5}$$

$$(3 \;\; 6) \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 3 \cdot 1 + 6(-1) = \underline{-3}$$

$$(3 \;\; 6) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3 \cdot 0 + 6 \cdot 1 = \underline{6}$$

Conclusion:

$$\underbrace{\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}}_{3\times 2} \cdot \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}}_{2 \times 2} = \begin{pmatrix} 4 & -3 \\ 5 & -3 \\ 6 & -3 \end{pmatrix} \qquad 2 = 2$$

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}}_{2\times 2} \cdot \underbrace{\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}}_{3\times 2} = \text{not defined} \qquad 2 \neq 3$$

Note: (1) $A \cdot B \neq B \cdot A$!

(2) $A \cdot B$ defined if $\boxed{\# \text{ cols in } A = \# \text{ rows in } B}$

SEE [LN] SECTION 2.1

2.1 Matrix Operations

Matrix Notation:

Two ways to denote $m \times n$ matrix A:

In terms of the *columns* of A:

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$$

In terms of the *entries* of A:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Main diagonal entries: $a_{11}, a_{22}, \ldots, a_{mm}$

Zero matrix:

$$0 = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

THEOREM 1

Let A, B, and C be matrices of the same size, and let r and s be scalars. Then

a. $A+B = B+A$	d. $r(A+B) = rA+rB$
b. $(A+B)+C = A+(B+C)$	e. $(r+s)A = rA+sA$
c. $A+0 = A$	f. $r(sA) = (rs)A$

Matrix Multiplication

Multiplying B and $\mathbf{x}$ transforms $\mathbf{x}$ into the vector $B\mathbf{x}$. In turn, if we multiply A and $B\mathbf{x}$, we transform $B\mathbf{x}$ into $A(B\mathbf{x})$. So $A(B\mathbf{x})$ is the composition of two mappings.

Define the product AB so that $A(B\mathbf{x}) = (AB)\mathbf{x}$.

Suppose A is $m \times n$ and B is $n \times p$ where

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}.$$

Then

$$B\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_p\mathbf{b}_p$$

and

$$A(B\mathbf{x}) = A(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_p\mathbf{b}_p)$$

$$= A(x_1\mathbf{b}_1) + A(x_2\mathbf{b}_2) + \cdots + A(x_p\mathbf{b}_p)$$

$$= x_1A\mathbf{b}_1 + x_2A\mathbf{b}_2 + \cdots + x_pA\mathbf{b}_p = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}.$$

Therefore,

$$A(B\mathbf{x}) = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}\mathbf{x}.$$

and by defining

$$\boxed{AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}}$$

we have $A(B\mathbf{x}) = (AB)\mathbf{x}$.

EXAMPLE: Compute AB where $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix}$.

Solution:

$$A\mathbf{b}_1 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix}, \qquad A\mathbf{b}_2 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ -24 \\ 6 \end{bmatrix} \qquad\qquad = \begin{bmatrix} 2 \\ 26 \\ -7 \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} -4 & 2 \\ -24 & 26 \\ 6 & -7 \end{bmatrix}$$

Note that $A\mathbf{b}_1$ is a linear combination of the columns of A and $A\mathbf{b}_2$ is a linear combination of the columns of A.

Each column of AB is a linear combination of the columns of A using weights from the corresponding columns of B.

EXAMPLE: If A is 4×3 and B is 3×2, then what are the sizes of AB and BA?

Solution:

$$AB = \underset{4 \times 3}{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}} \underset{3 \times 2}{\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}} = \underset{4 \times 2}{\begin{bmatrix} & \\ & \end{bmatrix}}$$

BA would be $\underset{3 \times 2}{\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}} \underset{4 \times 3}{\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}}$

which is not defined. $(2 \neq 4)$

If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$.

Row-Column Rule for Computing AB (alternate method)

The definition

$$\boxed{AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}}$$

is good for theoretical work.

When A and B have small sizes, the following method is more efficient when working by hand.

If AB is defined, let $(AB)_{ij}$ denote the entry in the ith row and jth column of AB. Then

column j in B

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

row i in A

$$\begin{bmatrix} & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{bmatrix}\begin{bmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \\ & b_{nj} & \end{bmatrix}$$

$$= \begin{bmatrix} & & \\ & (AB)_{ij} & \\ & & \end{bmatrix}$$

entry in position (i,j)

EXAMPLE $A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix}$. Compute AB, if it is defined.

2×3 3×2

Solution: Since A is 2×3 and B is 3×2, then AB is defined and AB is __2__ × __2__.

$$AB = \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{6} \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{2} & -3 \\ \mathbf{0} & 1 \\ \mathbf{4} & -7 \end{bmatrix} = \begin{bmatrix} \mathbf{28} & \blacksquare \\ \blacksquare & \blacksquare \end{bmatrix}$$

$2 \cdot 2 + 3 \cdot 0 + 6 \cdot 4$

$$\begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{6} \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & \mathbf{-3} \\ 0 & \mathbf{1} \\ 4 & \mathbf{-7} \end{bmatrix} = \begin{bmatrix} 28 & \mathbf{-45} \\ \blacksquare & \blacksquare \end{bmatrix}$$

$2 \cdot (-3) + 3 \cdot 1 + 6 \cdot (-7)$

$$\begin{bmatrix} 2 & 3 & 6 \\ \mathbf{-1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{2} & -3 \\ \mathbf{0} & 1 \\ \mathbf{4} & -7 \end{bmatrix} = \begin{bmatrix} 28 & -45 \\ \mathbf{2} & \blacksquare \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 6 \\ \mathbf{-1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} 2 & \mathbf{-3} \\ 0 & \mathbf{1} \\ 4 & \mathbf{-7} \end{bmatrix} = \begin{bmatrix} 28 & -45 \\ 2 & \mathbf{-4} \end{bmatrix}$$

So $AB = \begin{bmatrix} 28 & -45 \\ 2 & -4 \end{bmatrix}$.

$$I_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

THEOREM 2

Let A be $m \times n$ and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B+C) = AB + AC$ (left - distributive law)

c. $(B+C)A = BA + CA$ (right-distributive law)

d. $r(AB) = (rA)B = A(rB)$

for any scalar r

e. $I_m A = A = A I_n$ (identity for matrix multiplication)

I = identity matrix = "one" for matrices

WARNINGS

Properties above are analogous to properties of real numbers. But **NOT ALL** real number properties correspond to matrix properties.

1. It is not the case that AB always equal BA. (see Example 7, page 114)
2. Even if $AB = AC$, then B may not equal C. (see Exercise 10, page 116)
3. It is possible for $AB = 0$ even if $A \neq 0$ and $B \neq 0$. (see Exercise 12, page 116)

$AB = AC \Rightarrow AB - AC = 0 \Rightarrow A \cdot (B - C) = 0$

For numbers: If $A \neq 0$, then $B - C = 0 \Rightarrow B = C$

For matrices: Even if $A \neq 0$, $B \neq C$ can happen

$m\times n \quad m\times n$

ok if $m=n$

$A^2 = A \cdot A$

$A^3 = A \cdot A \cdot A$

Powers of A

Defined if A is square
(# cols = # rows)

$$A^k = \underbrace{A \cdots A}_{k}$$

EXAMPLE:

$$\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 9 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 21 & 8 \end{bmatrix}$$

If A is $m \times n$, the **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 8 \\ 7 & 6 & 5 & 4 & 3 \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} 1 & 6 & 7 \\ 2 & 7 & 6 \\ 3 & 8 & 5 \\ 4 & 9 & 4 \\ 5 & 8 & 3 \end{bmatrix}$$

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}$. Compute AB, $(AB)^T$, A^TB^T and B^TA^T.

Solution:

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 10 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 1 & 1 \\ 4 & 10 \end{bmatrix}$$

$$A^TB^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 10 \\ 2 & 0 & -4 \\ 2 & 1 & 4 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 10 \end{bmatrix}$$

<u>In general:</u> $(AB)^T = B^TA^T$

THEOREM 3

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$ (I.e., the transpose of A^T is A)

b. $(A+B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$ (I.e. the transpose of a product of matrices equals the product of their transposes in reverse order.)

EXAMPLE: Prove that $(ABC)^T =$ $\underline{C^T B^T A^T}$.

Solution: By Theorem 3d,

$$(ABC)^T = ((AB)C)^T = C^T(\qquad)^T$$

$$= C^T(\qquad) = \underline{\qquad\qquad}.$$

DETERMINANTS

For every square matrix A (# cols = # rows), we can compute the determinant

$$\det(A) = |A|$$

It is a numerical value.

Eks: A 2×2

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \underline{ad - bc}$$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow |A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1\cdot 4 - 2\cdot 3 = 4 - 6 = \underline{\underline{-2}}$$

SEE [LN] SECTION 3.1-3.2

When A is $n\times n$-matrix with $n \geq 3$, there are two methods for computing $|A|$:

1. Cofactor expansion
2. Gaussian elimination / row operations

} See [LN] for details

Defn: An inverse of A is a matrix B (also $n\times n$) such that

$$A\cdot B = B\cdot A = I_n$$

Facts: i) If an inverse of A exists, it is unique (and is usually called A^{-1}, not B)

ii) Even if $A \neq 0$ (zero matrix), it is not sure that A^{-1} exists

Theorem:

$$\boxed{A^{-1} \text{ exists} \iff \det(A) \neq 0}$$

This is the main property of the determinant.

In other words:

If $|A| \neq 0$, A^{-1} exists

If $|A| = 0$, A^{-1} does not exist

Case $n=2$: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

i) $|A| = \det(A) = ad - bc$

ii) $A^{-1} = \begin{cases} \dfrac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, & \text{if } ad-bc \neq 0 \\ \text{does not exist}, & \text{if } ad-bc = 0 \end{cases}$

Ex: Use of determinants and inverses

Linear system: $\begin{cases} x - y = 7 \\ \quad\;\; y = 13 \end{cases} \Rightarrow$ Matrix form: $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 13 \end{pmatrix}$

$A \cdot \underline{x} = \underline{b}$.

Solution of $A\underline{x} = \underline{b}$:

$$A\underline{x} = \underline{b}$$

$$A^{-1} \cdot (A\underline{x}) = A^{-1} \cdot \underline{b}$$

$$(A^{-1}A)\,\underline{x} = A^{-1}\underline{b}$$

$$I\,\underline{x} = A^{-1}\underline{b}$$

$$\underline{x} = A^{-1} \cdot \underline{b}$$

If A^{-1} exists

$\Updownarrow$

$|A| \neq 0$

$$x = A^{-1} \cdot \underline{b} = \frac{1}{1 \cdot 1 - (-1) \cdot 0} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 13 \end{pmatrix}$$

$$\underline{x} = \frac{1}{1} \cdot \begin{pmatrix} 7+13 \\ 13 \end{pmatrix} = \begin{pmatrix} 20 \\ 13 \end{pmatrix}$$

(since $|A| = 1 \cdot 1 - (-1) \cdot 0 = 1 \neq 0$)

3.1 Introduction to Determinants

Notation: A_{ij} is the matrix obtained from matrix A by deleting the ith row and jth column of A.

EXAMPLE:

$$A = \begin{array}{cc} & \begin{array}{cccc} & & 3 & \end{array} \\ \begin{array}{c} \\ 2 \\ \\ \\ \end{array} & \begin{bmatrix} 1 & 2 & \cancel{3} & 4 \\ \cancel{5} & \cancel{6} & \cancel{7} & \cancel{8} \\ 9 & 10 & \cancel{11} & 12 \\ 13 & 14 & \cancel{15} & 16 \end{bmatrix} \end{array} \qquad A_{23} = \begin{bmatrix} 1 & 2 & 4 \\ 9 & 10 & 12 \\ 13 & 14 & 16 \end{bmatrix}$$

Recall that $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ and we let $\det[a] = a$.

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is given by

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + \cdots + (-1)^{1+n}a_{1n}\det A_{1n}$$

$$= \sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det A_{1j}$$

cofactor expansion (first row)

EXAMPLE: Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

Solution

$$\det A = 1\det\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} - 2\det\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} + 0\det\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$$

$$= \underline{1 \cdot (-1) - 2 \cdot (3-4) + 0} = \underline{1}$$

Common notation: $\det\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}$.

So

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix}$$

The **(i, j)-cofactor** of A is the number C_{ij} where $C_{ij} = (-1)^{i+j} \det A_{ij}$.

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1C_{11} + 2C_{12} + 0C_{13}$$

(cofactor expansion across row 1)

THEOREM 1 The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column:

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \qquad \text{(expansion across row } i\text{)}$$

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \qquad \text{(expansion down column } j\text{)}$$

Use a matrix of signs to determine $(-1)^{i+j}$

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

EXAMPLE: Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$ using cofactor expansion down column 3.

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 1.$$

EXAMPLE: Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{bmatrix}$

Solution

cofactor expansion

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & 1 & 5 \\ 0 & 2 & 1 \\ 0 & 3 & 5 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 & 4 \\ 0 & 2 & 1 \\ 0 & 3 & 5 \end{vmatrix} + 0\begin{vmatrix} 2 & 3 & 4 \\ 2 & 1 & 5 \\ 0 & 3 & 5 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 & 4 \\ 2 & 1 & 5 \\ 0 & 2 & 1 \end{vmatrix}$$

new cofactor expansion

$$= 1 \cdot 2\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 14$$

Method of cofactor expansion is not practical for large matrices - see Numerical Note on page 190.

Triangular Matrices:

$$\begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ 0 & 0 & \ddots & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & \ddots & 0 & 0 \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \end{bmatrix}$$

(upper triangular) (lower triangular)

THEOREM 2: If A is a triangular matrix, then $\det A$ is the product of the main diagonal entries of A.

EXAMPLE:

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 4 \end{vmatrix} = \underline{\quad 2 \cdot 1 \cdot (-3) \cdot 4 \quad} = -24$$

3.2 Properties of Determinants

THEOREM 3 Let A be a square matrix.

a. If a multiple of one row of A is added to another row of A to produce a matrix B, then $\det A = \det B$.

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

EXAMPLE: Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix}$.

Solution

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix} = 5\begin{vmatrix} 1 & 3 & 4 \\ 2 & 6 & 10 \\ 2 & 7 & 11 \end{vmatrix} \overset{-2}{=} 5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 2 & 7 & 11 \end{vmatrix} {\scriptstyle -2}$$

$$= 5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = -5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{vmatrix} = \underline{-5 \cdot 1 \cdot 1 \cdot 2} = \underline{-10}.$$

Theorem 3(c) indicates that $\begin{vmatrix} * & * & * \\ -2k & 5k & 4k \\ * & * & * \end{vmatrix} = k \begin{vmatrix} * & * & * \\ -2 & 5 & 4 \\ * & * & * \end{vmatrix}$.

EXAMPLE: Compute $\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} :2 = 2 \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} \overset{-5,\ -7}{=} 2 \begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & -8 & -11 \end{vmatrix} :(-4)$$

$$= 2(-4) \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -8 & -11 \end{vmatrix} \overset{8}{=} 2(-4) \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{vmatrix}$$

$$= 2(-4)(1)(1)(5) = -40$$

EXAMPLE: Compute $\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix}$ using a combination of row reduction and cofactor expansion.

Solution
$$\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix} = -2\begin{vmatrix} 2 & 3 & 1 \\ 4 & 7 & 3 \\ 1 & 2 & 4 \end{vmatrix} = -2\begin{vmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \\ 0 & 1 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -7 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -7 \\ 0 & 0 & -6 \end{vmatrix} = -2(1)(-1)(-6) = -12.$$

Suppose A has been reduced to $U = \begin{bmatrix} \blacksquare & * & * & \cdots & * \\ 0 & \blacksquare & * & \cdots & * \\ 0 & 0 & \blacksquare & \cdots & * \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix}$ by row replacements and row interchanges, then

$$\det A = \begin{cases} (-1)^r \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

THEOREM 4 A square matrix is invertible if and only if $\det A \neq 0$.

THEOREM 5 If A is an $n \times n$ matrix, then $\det A^T = \det A$.

Partial proof (2×2 case)

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \qquad \text{and}$$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc$$

$$\Rightarrow \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

(3 × 3 case)

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\det\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = a\begin{vmatrix} e & h \\ f & i \end{vmatrix} - b\begin{vmatrix} d & g \\ f & i \end{vmatrix} + c\begin{vmatrix} d & g \\ e & h \end{vmatrix}$$

$$\Rightarrow \det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \det\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}.$$

Implications of Theorem 5?

Theorem 3 still holds if the word *row* is replaced

with column.

THEOREM 6 (Multiplicative Property)

For $n \times n$ matrices A and B, $\det(AB) = (\det A)(\det B)$.

EXAMPLE: Compute $\det A^3$ if $\det A = 5$.

Solution: $\det A^3 = \det(AAA) = (\det A)(\det A)(\det A)$

$= \underline{\hspace{4cm}} = \underline{\hspace{2cm}}.$

EXAMPLE: For $n \times n$ matrices A and B, show that A is singular if $\det B \neq 0$ and $\det AB = 0$.

Solution: *Since*

$$(\det A)(\det B) = \det AB = 0$$

and

$$\det B \neq 0,$$

then $\det A = 0$. Therefore A is singular.

A singular means

det A = 0