Chapter 15

15.1

1. (a), (c), (d), and (f) are linear, (b) and (e) are nonlinear.

2. If $a_{ij} = 1$ for all $i \neq j$ and $a_{ii} = 0$ for $i = 1, 2, 3, 4$, then the system is $\begin{cases} x_2 + x_3 + x_4 = b_1 \\ x_1 + x_3 + x_4 = b_2 \\ x_1 + x_2 + x_4 = b_3 \\ x_1 + x_2 + x_3 = b_4 \end{cases}$.

$x_1 = -\frac{2}{3}b_1 + \frac{1}{3}(b_2+b_3+b_4), x_2 = -\frac{2}{3}b_2 + \frac{1}{3}(b_1+b_3+b_4), x_3 = -\frac{2}{3}b_3 + \frac{1}{3}(b_1+b_2+b_4), x_4 = -\frac{2}{3}b_4 + \frac{1}{3}(b_1+b_2+b_3)$

3. $2x_1 + 3x_2 + 4x_3 = 1$
$3x_1 + 4x_2 + 5x_3 = 2$
$4x_1 + 5x_2 + 6x_3 = 3$

4. $x_1 = \frac{1}{4}x_2 + 100, x_2 = 2x_3 + 80, x_3 = \frac{1}{2}x_1$. Solution: $x_1 = 160, x_2 = 240, x_3 = 80$.

5. (a) The commodity bundle owned by individual j. (b) $a_{i1} + a_{i2} + \cdots + a_{in}$ is the total stock of commodity i. The first case is when $i = 1$. (c) $p_1 a_{1j} + p_2 a_{2j} + \cdots + p_m a_{mj}$

6. The equation system is: $\begin{cases} 0.712y - c = -95.05 \\ 0.158x - s + 0.158c = 34.30 \\ x - y - s + c = 0 \\ x = 93.53 \end{cases}$.

Solution : $x = 93.53, y \approx 482.11, s \approx 49.73, c \approx 438.31$

15.2

1. $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **2.** $\mathbf{A}+\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 7 & 5 \end{pmatrix}, 3\mathbf{A} = \begin{pmatrix} 0 & 3 \\ 6 & 9 \end{pmatrix}$

3. $u = 3$ and $v = -2$. (Equating the elements in row 1 and column 3 gives $u = 3$. Then, equating those in row 2 and column 3 gives $u - v = 5$ and so $v = -2$. The other elements then need to be checked, but this is obvious.)

4. $\mathbf{A}+\mathbf{B} = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 4 & 16 \end{pmatrix}, \mathbf{A}-\mathbf{B} = \begin{pmatrix} -1 & 2 & -6 \\ 2 & 2 & -2 \end{pmatrix}$, and $5\mathbf{A} - 3\mathbf{B} = \begin{pmatrix} -3 & 8 & -20 \\ 10 & 12 & 8 \end{pmatrix}$

15.3

1. (a) $\mathbf{AB} = \begin{pmatrix} -2 & -10 \\ -2 & 17 \end{pmatrix}$ and $\mathbf{BA} = \begin{pmatrix} 12 & 6 \\ 15 & 3 \end{pmatrix}$ (b) $\mathbf{AB} = \begin{pmatrix} 26 & 3 \\ 6 & -22 \end{pmatrix}$ and $\mathbf{BA} = \begin{pmatrix} 14 & 6 & -12 \\ 35 & 12 & 4 \\ 3 & 3 & -22 \end{pmatrix}$

(c) $\mathbf{AB} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & -6 \\ 0 & -8 & 12 \end{pmatrix}$ and $\mathbf{BA} = (16)$, a 1×1 matrix. (d) $\mathbf{AB}$ is not defined. $\mathbf{BA} = \begin{pmatrix} -1 & 4 \\ 3 & 4 \\ 4 & 8 \end{pmatrix}$

2. (i) $\begin{pmatrix} -1 & 15 \\ -6 & -13 \end{pmatrix}$ (ii) and (iii): $\mathbf{AB} = \mathbf{C}(\mathbf{AB}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, since $\mathbf{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

3. $\mathbf{A}+\mathbf{B} = \begin{pmatrix} 4 & 1 & -1 \\ 9 & 2 & 7 \\ 3 & -1 & 4 \end{pmatrix}, \mathbf{A}-\mathbf{B} = \begin{pmatrix} -2 & 3 & -5 \\ 1 & -2 & -3 \\ -1 & -1 & -2 \end{pmatrix}, \mathbf{AB} = \begin{pmatrix} 5 & 3 & 3 \\ 19 & -5 & 16 \\ 1 & -3 & 0 \end{pmatrix}, \mathbf{BA} = \begin{pmatrix} 0 & 4 & -9 \\ 19 & 3 & -3 \\ 5 & 1 & -3 \end{pmatrix}$,

$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) = \begin{pmatrix} 23 & 8 & 25 \\ 92 & -28 & 76 \\ 4 & -8 & -4 \end{pmatrix}$

4. (a) $\begin{pmatrix} 1 & 1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & -3 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

5. (a) The product **AB** is only defined if **B** has n rows. And **BA** is only defined if **B** has m columns. So **B** must be an $n \times m$ matrix. (b) $\mathbf{B} = \begin{pmatrix} w-y & y \\ y & w \end{pmatrix}$, for arbitrary y, w.

6. $\mathbf{T}(\mathbf{Ts}) = \begin{pmatrix} 0.85 & 0.10 & 0.10 \\ 0.05 & 0.55 & 0.05 \\ 0.10 & 0.35 & 0.85 \end{pmatrix}\begin{pmatrix} 0.25 \\ 0.35 \\ 0.40 \end{pmatrix} = \begin{pmatrix} 0.2875 \\ 0.2250 \\ 0.4875 \end{pmatrix}$

15.4

1. $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC} = \begin{pmatrix} 3 & 2 & 6 & 2 \\ 7 & 4 & 14 & 6 \end{pmatrix}$ **2.** $(ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz)$ (a 1×1 matrix)

3. It is straightforward to show that $(\mathbf{AB})\mathbf{C}$ and $\mathbf{A}(\mathbf{BC})$ are both equal to the 2×2 matrix $\mathbf{D} = (d_{ij})_{2\times 2}$ with $d_{ij} = a_{i1}b_{11}c_{1j} + a_{i1}b_{12}c_{2j} + a_{i2}b_{21}c_{1j} + a_{i2}b_{22}c_{2j}$ for $i = 1, 2$ and $j = 1, 2$.

4. (a) $\begin{pmatrix} 5 & 3 & 1 \\ 2 & 0 & 9 \\ 1 & 3 & 3 \end{pmatrix}$ (b) $(1, 2, -3)$

5. Equality in (a) as well as in (b) if and only if $\mathbf{AB} = \mathbf{BA}$. ($(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \mathbf{A}^2 - \mathbf{AB} + \mathbf{BA} - \mathbf{B}^2 \neq \mathbf{A}^2 - \mathbf{B}^2$ unless $\mathbf{AB} = \mathbf{BA}$. The other case is similar.)

6. (a) Direct verification by matrix multiplication. (b) $\mathbf{AA} = (\mathbf{AB})\mathbf{A} = \mathbf{A}(\mathbf{BA}) = \mathbf{AB} = \mathbf{A}$, so **A** is idempotent. Then just interchange **A** and **B** to show that **B** is idempotent. (c) As the induction hypothesis, suppose that $\mathbf{A}^k = \mathbf{A}$, which is true for $k = 1$. Then $\mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{A} = \mathbf{AA} = \mathbf{A}$, which completes the proof by induction.

7. (a) Direct verification. (b) $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ (c) See SM.

15.5

1. $\mathbf{A}' = \begin{pmatrix} 3 & -1 \\ 5 & 2 \\ 8 & 6 \\ 3 & 2 \end{pmatrix}$, $\mathbf{B}' = (0, 1, -1, 2)$, $\mathbf{C}' = \begin{pmatrix} 1 \\ 5 \\ 0 \\ -1 \end{pmatrix}$

2. $\mathbf{A}' = \begin{pmatrix} 3 & -1 \\ 2 & 5 \end{pmatrix}$, $\mathbf{B}' = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}$, $(\mathbf{A}+\mathbf{B})' = \begin{pmatrix} 3 & 1 \\ 4 & 7 \end{pmatrix}$, $(\alpha\mathbf{A})' = \begin{pmatrix} -6 & 2 \\ -4 & -10 \end{pmatrix}$, $\mathbf{AB} = \begin{pmatrix} 4 & 10 \\ 10 & 8 \end{pmatrix}$,

$(\mathbf{AB})' = \begin{pmatrix} 4 & 10 \\ 10 & 8 \end{pmatrix} = \mathbf{B}'\mathbf{A}'$, and $\mathbf{A}'\mathbf{B}' = \begin{pmatrix} -2 & 4 \\ 10 & 14 \end{pmatrix}$. Verifying the rules in (2) is now very easy.

3. Equation (1) implies that $\mathbf{A} = \mathbf{A}'$ and $\mathbf{B} = \mathbf{B}'$.

4. Symmetry requires $a^2 - 1 = a + 1$ and $a^2 + 4 = 4a$. The second equation has the unique root $a = 2$, which also satisfies the first equation.

5. No! For example: $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$.

6. $(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3)' = (\mathbf{A}_1(\mathbf{A}_2\mathbf{A}_3))' = (\mathbf{A}_2\mathbf{A}_3)'\mathbf{A}_1' = (\mathbf{A}_3'\mathbf{A}_2')\mathbf{A}_1' = \mathbf{A}_3'\mathbf{A}_2'\mathbf{A}_1'$. For the general case use induction.

7. (a) Direct verification. (b) $\begin{pmatrix} p & q \\ -q & p \end{pmatrix}\begin{pmatrix} p & -q \\ q & p \end{pmatrix} = \begin{pmatrix} p^2+q^2 & 0 \\ 0 & p^2+q^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ iff $p^2 + q^2 = 1$.

(c) If $\mathbf{P}'\mathbf{P} = \mathbf{Q}'\mathbf{Q} = \mathbf{I}_n$, then $(\mathbf{PQ})'(\mathbf{PQ}) = (\mathbf{Q}'\mathbf{P}')(\mathbf{PQ}) = \mathbf{Q}'(\mathbf{P}'\mathbf{P})\mathbf{Q} = \mathbf{Q}'\mathbf{I}_n\mathbf{Q} = \mathbf{Q}'\mathbf{Q} = \mathbf{I}_n$.

8. (a) $\mathbf{TS} = \begin{pmatrix} p^3+p^2q & 2p^2q+2pq^2 & pq^2+q^3 \\ \frac{1}{2}p^3+\frac{1}{2}p^2+\frac{1}{2}p^2q & p^2q+pq+pq^2 & \frac{1}{2}pq^2+\frac{1}{2}q^2+\frac{1}{2}q^3 \\ p^3+p^2q & 2p^2q+2pq^2 & pq^2+q^3 \end{pmatrix} = \mathbf{S}$ because $p+q=1$.

A similar argument shows that $\mathbf{T}^2 = \frac{1}{2}\mathbf{T} + \frac{1}{2}\mathbf{S}$. To derive the formula for $\mathbf{T}^3$, don't look at individual elements.

(b) The appropriate formula is $\mathbf{T}^n = 2^{1-n}\mathbf{T} + (1-2^{1-n})\mathbf{S}$.

15.6

1. (a) Gaussian elimination yields

$$\begin{pmatrix} 1 & 1 & 3 \\ 3 & 5 & 5 \end{pmatrix} \begin{matrix} -3 \\ \leftarrow \end{matrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & -4 \end{pmatrix} 1/2 \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & -2 \end{pmatrix} \begin{matrix} \leftarrow \\ -1 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \end{pmatrix}$$

The solution is therefore $x_1 = 5$, $x_2 = -2$. (b) Gaussian elimination yields

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 1 & -1 & 1 & 5 \\ 2 & 3 & -1 & 1 \end{pmatrix} \begin{matrix} -1 \quad -2 \\ \leftarrow \\ \leftarrow \end{matrix} \sim \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & -3 & 0 & 1 \\ 0 & -1 & -3 & -7 \end{pmatrix} -1/3 \sim \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & -1/3 \\ 0 & -1 & -3 & -7 \end{pmatrix} \begin{matrix} \leftarrow \\ 1 \quad -2 \\ \leftarrow \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 14/3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & -3 & -22/3 \end{pmatrix} -1/3 \sim \begin{pmatrix} 1 & 0 & 1 & 14/3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 22/9 \end{pmatrix} \begin{matrix} \leftarrow \\ \\ -1 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 & 20/9 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 22/9 \end{pmatrix}$$

The solution is therefore: $x_1 = 20/9$, $x_2 = -1/3$, $x_3 = 22/9$

(c) Solution: $x_1 = (2/5)s$, $x_2 = (3/5)s$, $x_3 = s$, with s an arbitrary real number.

2. Gaussian elimination yields eventually: $\begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -3/2 & -1/2 \\ 0 & 0 & a+5/2 & b-1/2 \end{pmatrix}$.

For any z, the first two equations imply that $y = -\frac{1}{2} + \frac{3}{2}z$ and $x = 1 - y + z = \frac{3}{2} - \frac{1}{2}z$. From the last equation we see that for $a \neq -\frac{5}{2}$, there is a unique solution with $z = (b - \frac{1}{2})/(a + \frac{5}{2})$. For $a = -\frac{5}{2}$, there are no solutions if $b \neq \frac{1}{2}$, but there is one degree of freedom if $b = \frac{1}{2}$ (with z arbitrary).

3. For $c = 1$ and for $c = -2/5$ the solution is $x = 2c^2 - 1 + t$, $y = s$, $z = t$, $w = 1 - c^2 - 2s - 2t$, for arbitrary s and t. For other values of c there are no solutions.

4. (a) Move the first row down to row number three and use Gaussian elimination. There is a unique solution iff $a \neq 3/4$. (b) If $b_1 \neq \frac{1}{4}b_3$ there is no solution. If $b_1 = \frac{1}{4}b_3$ there are an infinite number of solutions. In fact, $x = -2b_2 + b_3 - 5t$, $y = \frac{3}{2}b_2 - \frac{1}{2}b_3 + 2t$, $z = t$, with $t \in \mathbb{R}$.

15.7

1. $\mathbf{a}+\mathbf{b} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, $\mathbf{a}-\mathbf{b} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$, $2\mathbf{a}+3\mathbf{b} = \begin{pmatrix} 13 \\ 10 \end{pmatrix}$, and $-5\mathbf{a}+2\mathbf{b} = \begin{pmatrix} -4 \\ 13 \end{pmatrix}$

2. $\mathbf{a}+\mathbf{b}+\mathbf{c} = (-1, 6, -4)$, $\mathbf{a}-2\mathbf{b}+2\mathbf{c} = (-3, 10, 2)$, $3\mathbf{a}+2\mathbf{b}-3\mathbf{c} = (9, -6, 9)$

3. $x = 3$, $y = -3$, $z = -4$ **4.** (a) $x_i = 0$ for all i. (b) Nothing, because $0 \cdot \mathbf{x} = \mathbf{0}$ for all $\mathbf{x}$.

5. $(4, -11) = 3(2, -1) - 2(1, 4)$ **6.** $4\mathbf{x} - 2\mathbf{x} = 7\mathbf{a} + 8\mathbf{b} - \mathbf{a}$, so $2\mathbf{x} = 6\mathbf{a} + 8\mathbf{b}$, and $\mathbf{x} = 3\mathbf{a} + 4\mathbf{b}$.

7. $\mathbf{a}\cdot\mathbf{a} = 5$, $\mathbf{a}\cdot\mathbf{b} = 2$, and $\mathbf{a}\cdot(\mathbf{a}+\mathbf{b}) = 7$. We see that $\mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot(\mathbf{a}+\mathbf{b})$.

8. The inner product of the two vectors is $x^2 + (x-1)x + 3 \cdot 3x = x^2 + x^2 - x + 9x = 2x^2 + 8x = 2x(x+4)$, which is 0 for $x = 0$ and $x = -4$.

9. $\mathbf{x} = (5, 7, 12)$, $\mathbf{u} = (20, 18, 25)$, $\mathbf{u}\cdot\mathbf{x} = 526$

Chapter 16

16.1

1. (a) $3\cdot 6-2\cdot 0=18$ (b) $ab-ba=0$ (c) $(a+b)^2-(a-b)^2=4ab$ (d) $3^t2^{t-1}-3^{t-1}2^t=3^{t-1}2^{t-1}(3-2)=6^{t-1}$

2. See Fig. A16.1.2. The shaded parallelogram has area $3\cdot 6=18=\begin{vmatrix}3 & 0\\ 2 & 6\end{vmatrix}$.

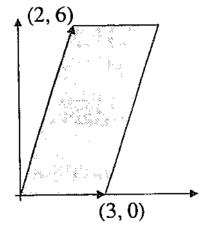

Figure A16.1.2

3. (a) $x=11/5$ and $y=-7/5$ (b) $x=4$ and $y=-1$ (c) $x=\dfrac{a+2b}{a^2+b^2}$, $y=\dfrac{2a-b}{a^2+b^2}$, $(a^2+b^2\neq 0)$

4. The matrix product is $\mathbf{AB}=\begin{pmatrix}a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22}\\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22}\end{pmatrix}$, implying that $|\mathbf{AB}|=(a_{11}b_{11}+a_{12}b_{21})(a_{21}b_{12}+a_{22}b_{22})-(a_{11}b_{12}+a_{12}b_{22})(a_{21}b_{11}+a_{22}b_{21})$. On the other hand, $|\mathbf{A}||\mathbf{B}|=(a_{11}a_{22}-a_{12}a_{21})(b_{11}b_{22}-b_{12}b_{21})$. A tedious process of expanding each expression, then canceling four terms in the expression of $|\mathbf{A}||\mathbf{B}|$, reveals that the two expressions are equal.

5. If $\mathbf{A}=\mathbf{B}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, then $|\mathbf{A}+\mathbf{B}|=4$, whereas $|\mathbf{A}|+|\mathbf{B}|=2$. ($\mathbf{A}$ and $\mathbf{B}$ can be chosen almost arbitrarily.)

6. We write the system as $\begin{cases} Y-C=I_0+G_0\\ -bY+C=a\end{cases}$. Then Cramer's rule yields

$$Y=\frac{\begin{vmatrix}I_0+G_0 & -1\\ a & 1\end{vmatrix}}{\begin{vmatrix}1 & -1\\ -b & 1\end{vmatrix}}=\frac{a+I_0+G_0}{1-b},\qquad C=\frac{\begin{vmatrix}1 & I_0+G_0\\ -b & a\end{vmatrix}}{\begin{vmatrix}1 & -1\\ -b & 1\end{vmatrix}}=\frac{a+b(I_0+G_0)}{1-b}$$

The expression for Y is most easily found by substituting the second equation into the first, and then solving for Y. Then use $C=a+bY$ to find C.

7. (a) $X_1=M_2$ because nation 1's exports are nation 2's imports. Similarly, $X_2=M_1$.
(b) Substituting for X_1, X_2, M_1, M_2, C_1, and C_2 gives: (i) $Y_1(1-c_1+m_1)-m_2Y_2=A_1$;
(ii) $Y_2(1-c_2+m_2)-m_1Y_1=A_2$. Using Cramer's rule with $D=(1-c_2+m_2)(1-c_1+m_1)-m_1m_2$ yields

$$Y_1=[A_2m_2+A_1(1-c_2+m_2)]/D,\qquad Y_2=[A_1m_1+A_2(1-c_1+m_1)]/D$$

(c) Y_2 increases when A_1 increases.

16.2

1. (a) -2 (b) -2 (c) adf (d) $e(ad-bc)$

7. $\mathbf{X}'\mathbf{X} = \begin{pmatrix} 4 & 3 & 2 \\ 3 & 5 & 1 \\ 2 & 1 & 2 \end{pmatrix}$ and $|\mathbf{X}'\mathbf{X}| = 10$

8. By Sarrus's rule, for example, $|\mathbf{A}_a| = a(a^2+1) + 4 + 4 - 4(a^2+1) - a - 4 = a^2(a-4)$, so $|\mathbf{A}_1| = -3$ and $|\mathbf{A}_1^6| = |\mathbf{A}_1|^6 = (-3)^6 = 729$. (If you don't use rule (16.4.1), but try to find $\mathbf{A}_1^6$ first, you are in great trouble.)

9. Because $\mathbf{P}'\mathbf{P} = \mathbf{I}_n$, it follows from (16.4.1) and (16.3.4) that $|\mathbf{P}'||\mathbf{P}| = |\mathbf{I}_n| = 1$. But $|\mathbf{P}'| = |\mathbf{P}|$ by rule B in Theorem 16.4.1, so $|\mathbf{P}|^2 = 1$. Hence, $|\mathbf{P}| = \pm 1$.

10. (a) Because $\mathbf{A}^2 = \mathbf{I}_n$ it follows from (16.4.1) that $|\mathbf{A}|^2 = |\mathbf{I}_n| = 1$, and so $|\mathbf{A}| = \pm 1$. (b) Direct verification by matrix multiplication. (c) Expand $(\mathbf{I}_n - \mathbf{A})(\mathbf{I}_n + \mathbf{A})$.

11. Let $\mathrm{A} = \begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix}$. Then compute $\mathbf{A}^2$ and recall (16.4.1).

12. Start by adding each of the last $n-1$ rows to the first row. Each element in the first row then becomes $na + b$. Factor this out of the determinant. Next, add the first row multiplied by $-a$ to all the other $n-1$ rows. The result is an upper triangular matrix whose diagonal elements are $1, b, b, \ldots, b$, with product equal to b^{n-1}. The conclusion follows easily.

16.5

1. (a) 2. (Subtract row 1 from both row 2 and row 3 to get a determinant whose first column has elements 1, 0, 0. Then expand by the first column.) (b) 30 (c) 0. (Columns 2 and 4 are proportional.)

2. In each of these cases we keep expanding by the last (remaining) column. The answers are: (a) $-abc$ (b) $abcd$ (c) $6 \cdot 4 \cdot 3 \cdot 5 \cdot 1 = 360$

16.6

1. Using (16.6.4): $\begin{pmatrix} 3 & 0 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1/3 & 0 \\ 2/3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. **2.** Multiply the two matrices to get $\mathbf{I}_3$.

3. $\mathbf{AB} = \begin{pmatrix} 1 & 0 & 0 \\ a+b & 2a+1/4+3b & 4a+3/2+2b \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$ iff $a+b = 4a+3/2+2b = 0$ and $2a+1/4+3b = 1$. This is true iff $a = -3/4$ and $b = 3/4$.

4. (a) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 3 & -4 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

(b) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 8 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (c) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

5. From $\mathbf{A}^3 = \mathbf{I}$, it follows that $\mathbf{A}^2\mathbf{A} = \mathbf{I}$, so $\mathbf{A}^{-1} = \mathbf{A}^2 = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$.

6. (a) $|\mathbf{A}| = 1$, $\mathbf{A}^2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$, $\mathbf{A}^3 = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}$. Direct verification yields $\mathbf{A}^3 - 2\mathbf{A}^2 + \mathbf{A} - \mathbf{I} = \mathbf{0}$.

(b) The equality shown in (a) is equivalent to $\mathbf{A}(\mathbf{A}-\mathbf{I})^2 = \mathbf{I}$, so $\mathbf{A}^{-1} = (\mathbf{A}-\mathbf{I})^2$.

(c) Choose $\mathbf{P} = (\mathbf{A}-\mathbf{I})^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, so that $\mathbf{A} = [(\mathbf{A}-\mathbf{I})^2]^{-1} = \mathbf{P}^2$. The matrix $-\mathbf{P}$ also works.

7. (a) $\mathbf{AA}' = \begin{pmatrix} 21 & 11 \\ 11 & 10 \end{pmatrix}$, $|\mathbf{AA}'| = 89$, and $(\mathbf{AA}')^{-1} = \frac{1}{89}\begin{pmatrix} 10 & -11 \\ -11 & 21 \end{pmatrix}$. (b) No, $\mathbf{AA}'$ is always symmetric by Example 15.5.3. Then $(\mathbf{AA}')^{-1}$ is symmetric by Note 2.

8. (a) $\mathbf{A}^2 = (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1}) = \mathbf{PD}(\mathbf{P}^{-1}\mathbf{P})\mathbf{DP}^{-1} = \mathbf{PDIDP}^{-1} = \mathbf{PD}^2\mathbf{P}^{-1}$.
(b) Suppose the formula is valid for $m = k$. Then $\mathbf{A}^{k+1} = \mathbf{AA}^k = \mathbf{PDP}^{-1}(\mathbf{PD}^k\mathbf{P}^{-1}) = \mathbf{PD}(\mathbf{P}^{-1}\mathbf{P})\mathbf{D}^k\mathbf{P}^{-1} = \mathbf{PDID}^k\mathbf{P}^{-1} = \mathbf{PDD}^k\mathbf{P}^{-1} = \mathbf{PD}^{k+1}\mathbf{P}^{-1}$.

9. $\mathbf{B}^2 + \mathbf{B} = \mathbf{I}$, $\mathbf{B}^3 - 2\mathbf{B} + \mathbf{I} = \mathbf{0}$, and $\mathbf{B}^{-1} = \mathbf{B} + \mathbf{I} = \begin{pmatrix} 1/2 & 5 \\ 1/4 & 1/2 \end{pmatrix}$.

10. (a) Let $\mathbf{B} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$. Then $\mathbf{A}^2 = (\mathbf{I}_m - \mathbf{B})(\mathbf{I}_m - \mathbf{B}) = \mathbf{I}_m - \mathbf{B} - \mathbf{B} + \mathbf{B}^2$. Here $\mathbf{B}^2 = (\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')(\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}') = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{X})(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' = \mathbf{B}$. Thus, $\mathbf{A}^2 = \mathbf{I}_m - \mathbf{B} - \mathbf{B} + \mathbf{B} = \mathbf{I}_m - \mathbf{B} = \mathbf{A}$.
(b) Direct verification.

11. (a) If $\mathbf{C}^2 + \mathbf{C} = \mathbf{I}$, then $\mathbf{C}(\mathbf{C} + \mathbf{I}) = \mathbf{I}$, and so $\mathbf{C}^{-1} = \mathbf{C} + \mathbf{I} = \mathbf{I} + \mathbf{C}$.
(b) Because $\mathbf{C}^2 = \mathbf{I} - \mathbf{C}$, it follows that $\mathbf{C}^3 = \mathbf{C}^2\mathbf{C} = (\mathbf{I} - \mathbf{C})\mathbf{C} = \mathbf{C} - \mathbf{C}^2 = \mathbf{C} - (\mathbf{I} - \mathbf{C}) = -\mathbf{I} + 2\mathbf{C}$. Moreover, $\mathbf{C}^4 = \mathbf{C}^3\mathbf{C} = (-\mathbf{I} + 2\mathbf{C})\mathbf{C} = -\mathbf{C} + 2\mathbf{C}^2 = -\mathbf{C} + 2(\mathbf{I} - \mathbf{C}) = 2\mathbf{I} - 3\mathbf{C}$.

16.7

1. (a) $\begin{pmatrix} -5/2 & 3/2 \\ 2 & -1 \end{pmatrix}$ (b) $\frac{1}{9}\begin{pmatrix} 1 & 4 & 2 \\ 2 & -1 & 4 \\ 4 & -2 & -1 \end{pmatrix}$ (c) The matrix has no inverse.

2. The inverse is $\dfrac{1}{|\mathbf{A}|}\begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = \dfrac{1}{72}\begin{pmatrix} -3 & 5 & 9 \\ 18 & -6 & 18 \\ 6 & 14 & -18 \end{pmatrix}$.

3. $(\mathbf{I} - \mathbf{A})^{-1} = \frac{5}{62}\begin{pmatrix} 18 & 16 & 10 \\ 2 & 19 & 8 \\ 4 & 7 & 16 \end{pmatrix}$

4. When $k = r$, the solution to the system is $x_1 = b^*_{1r}, x_2 = b^*_{2r}, \ldots, x_n = b^*_{nr}$.

5. (a) $\mathbf{A}^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{pmatrix}$ (c) There is no inverse.

16.8

1. (a) $x = 1$, $y = -2$, and $z = 2$ (b) $x = -3$, $y = 6$, $z = 5$, and $u = -5$

2. The determinant of the system is equal to -10, so the solution is unique. The determinants in (2) are

$$D_1 = \begin{vmatrix} b_1 & 1 & 0 \\ b_2 & -1 & 2 \\ b_3 & 3 & -1 \end{vmatrix}, \quad D_2 = \begin{vmatrix} 3 & b_1 & 0 \\ 1 & b_2 & 2 \\ 2 & b_3 & -1 \end{vmatrix}, \quad D_3 = \begin{vmatrix} 3 & 1 & b_1 \\ 1 & -1 & b_2 \\ 2 & 3 & b_3 \end{vmatrix}$$

Expanding each of these determinants by the column (b_1, b_2, b_3), we find that $D_1 = -5b_1 + b_2 + 2b_3$, $D_2 = 5b_1 - 3b_2 - 6b_3$, $D_3 = 5b_1 - 7b_2 - 4b_3$. Hence, $x_1 = \frac{1}{2}b_1 - \frac{1}{10}b_2 - \frac{1}{5}b_3$, $x_2 = -\frac{1}{2}b_1 + \frac{3}{10}b_2 + \frac{3}{5}b_3$, $x_3 = -\frac{1}{2}b_1 + \frac{7}{10}b_2 + \frac{2}{5}b_3$.

3. Show that the determinant of the coefficient matrix is equal to $-(a^3 + b^3 + c^3 - 3abc)$, and use Theorem 16.8.2.

16.9

1. (a) Let x and y denote total production in industries A and I, respectively. Then $x = \frac{1}{6}x + \frac{1}{4}y + 60$ and $y = \frac{1}{4}x + \frac{1}{4}y + 60$. So $\frac{5}{6}x - \frac{1}{4}y = 60$ and $-\frac{1}{4}x + \frac{3}{4}y = 60$. (b) The solution is $x = 320/3$ and $y = 1040/9$.

2. (a) No sector delivers to itself. (b) The total amount of good i needed to produce one unit of each good.
(c) This column vector gives the number of units of each good which are needed to produce one unit of good j.
(d) No meaningful economic interpretation. (The goods are usually measured in different units, so it is meaningless to add them together.)

3. $0.8x_1 - 0.3x_2 = 120$ and $-0.4x_1 + 0.9x_2 = 90$, with solution $x_1 = 225$ and $x_2 = 200$.