FORK1003 - Solutions for Final test in Linear Algebra

4th August 2010

Problem 1. Compute $-2A+5B$ when
$$A=\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix} \text{ and } B=\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}.$$

Solution. $-2A+5B=-2\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}+5\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}=\begin{pmatrix} -2 & 9 \\ 14 & 3 \end{pmatrix}$

Problem 2. Compute AB and BA, if possible, for the following:

(1) $A=\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$ and $B=\begin{pmatrix} -3 & 1 & 1 \end{pmatrix}$

(2) $A=\begin{pmatrix} 5 & -3 \\ 10 & 11 \end{pmatrix}$ and $B=\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

(3) $A=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} -1 & 2 \\ 3 & 2 \\ -1 & 2 \end{pmatrix}$

Solution. (1)
$$AB=\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}\begin{pmatrix} -3 & 1 & 1 \end{pmatrix}=\begin{pmatrix} -3 & 1 & 1 \\ 9 & -3 & -3 \\ -3 & 1 & 1 \end{pmatrix}$$
$$BA=\begin{pmatrix} -3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}=-5$$
(2)
AB is undefined
BA is undefined
(3)
$$AB=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 3 & 2 \\ -1 & 2 \end{pmatrix}=\begin{pmatrix} -2 & 4 \\ 3 & 2 \\ -1 & 2 \end{pmatrix}$$
BA is not defined

Problem 3. Compute the determinants
(a) $\begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & -13 \\ -2 & 12 \end{vmatrix}$

Solution. (a)
$$\begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix}=0$$
(b)
$$\begin{vmatrix} 2 & -13 \\ -2 & 12 \end{vmatrix}=-2$$

Problem 4. Write

$$5x_1 - 7x_2 = -2$$
$$7x_1 - 10x_2 = 1$$

as $A\mathbf{x} = \mathbf{b}$. Find A^{-1} and use this to solve the system of equations.

Solution. $A = \begin{pmatrix} 5 & -7 \\ 7 & -10 \end{pmatrix} \implies A^{-1} = \begin{pmatrix} 5 & -7 \\ 7 & -10 \end{pmatrix}^{-1} = \begin{pmatrix} 10 & -7 \\ 7 & -5 \end{pmatrix}$

$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} 10 & -7 \\ 7 & -5 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -27 \\ -19 \end{pmatrix}$

Problem 5. Compute the determinant of A by cofactor expansion along a suitable row and determine if the matrix is invertible.

(a) $A = \begin{pmatrix} 2 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$ (b) $A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 3 & -2 \\ -1 & 0 & 0 & 1 \end{pmatrix}$

Solution. (a)

$$\begin{vmatrix} 2 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix} = -2$$

(b)

$$\begin{vmatrix} 1 & -1 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 3 & -2 \\ -1 & 0 & 0 & 1 \end{vmatrix} = 15$$

Problem 6. Use Gauss elimination to solve the following linear system when $r = 1$:

$$x_1 + x_2 + x_3 = 6$$
$$x_1 - 2x_2 + 4x_3 = 3$$
$$x_1 - x_2 + rx_3 = 4$$

Are there any values of r such that the system is inconsistent? Are there any values of r such that the system has infinitely many solutions?

Solution. We form the augmented matrix of the linear system (for arbitrary r) and reduce it to echelon form using elementary row operations, and get

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 1 & -2 & 4 & 3 \\ 1 & -1 & r & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & r-3 & 0 \end{pmatrix}$$

For $r = 1$, this gives the reduced echelon form

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

hence we obtain $x_1 = 5$, $x_2 = 1$ and $x_3 = 0$. From the echelon form for arbitrary r, we see that the system has infinitely many solutions for $r = 3$, and that the system cannot be inconsistent.

Problem 7. Write the following system of linear equations as $A\mathbf{x} = \mathbf{b}$ and use Cramers rule to find x_2:

$$\begin{aligned} 2x_1 - x_2 + 2x_3 &= 0 \\ x_1 - 2x_2 - x_3 &= 3 \\ x_1 + x_2 - x_3 &= 0 \end{aligned}$$

Solution. $A = \begin{pmatrix} 2 & -1 & 2 \\ 1 & -2 & -1 \\ 1 & 1 & -1 \end{pmatrix}$, $A_{\mathbf{b},2} = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 3 & -1 \\ 1 & 0 & -1 \end{pmatrix}$

$$x_2 = \frac{\begin{vmatrix} 2 & 0 & 2 \\ 1 & 3 & -1 \\ 1 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & -1 \\ 1 & 1 & -1 \end{vmatrix}} = \frac{-12}{12} = -1$$

Problem 8. Find the inverse of the matrix

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

if it exists.

Solution. $\begin{pmatrix} 2 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ \frac{1}{2} & -1 & -\frac{1}{2} \end{pmatrix}$

Problem 9. Assume that

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 10 & 0 & -1 \end{pmatrix}.$$

Compute A^2. Is A invertible? If so, find the inverse of A without computing cofactors.

Solution. $A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 10 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 10 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies A^{-1} = A$