# Review of Lecture 2

## Matrix Algebra

Addition / subtraction
Scalar multiplication
Matrix multiplication
Powers
Transpose

## Vectors

Geometric representation
Linear combinations

## Inverses:

Formula for 2×2 - matrices

## Lecture 3:

## Determinants

A $n \times n$-matrix (square) $\rightsquigarrow$ $\det(A) = |A|$

gives a number

## Case of $2 \times 2$-matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$|A| = a_{11} a_{22} - a_{12} a_{21} = ad - bc$$

## 3.1 Introduction to Determinants

*Notation:* $A_{ij}$ is the matrix obtained from matrix $A$ by deleting the $i$th row and $j$th column of $A$.

**EXAMPLE:**

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ \cancel{5} & \cancel{6} & \cancel{7} & \cancel{8} \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \qquad A_{23} = \begin{bmatrix} 1 & 2 & 4 \\ 9 & 10 & 12 \\ 13 & 14 & 16 \end{bmatrix}$$

Recall that $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ and we let $\det[a] = a$.

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is given by

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + \cdots + (-1)^{1+n}a_{1n}\det A_{1n}$$

$$= \sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det A_{1j}$$

$= 1 \cdot \det A_{11} - 2 \cdot \det A_{12} + 3 \cdot \det A_{13}$

$- 4 \cdot \det(A_{14})$

**EXAMPLE:** Compute the determinant of $A = \begin{bmatrix} \cancel{1} & \cancel{2} & \cancel{0} \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

*Solution*

$$\det A = 1\det\overset{A_{11}}{\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}} - 2\det\overset{A_{12}}{\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}} + 0\det\overset{A_{13}}{\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}}$$

$$= \underline{1\cdot(-1\cdot 1 - 2\cdot 0) - 2(3\cdot 1 - 2\cdot 2)} = \underline{-1+2} = \underline{\underline{1}}$$

Common notation: $\det\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}$.

So

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix}$$

The **(i, j)-cofactor** of $A$ is the number $C_{ij}$ where
$C_{ij} = (-1)^{i+j}\det A_{ij}$.

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1\overset{+\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix}}{\overset{\parallel}{C_{11}}} + 2\overset{-\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}}{\overset{\parallel}{C_{12}}} + 0C_{13}$$

(cofactor expansion across row 1)

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0 \cdot C_{13} + 2 \cdot C_{23} + 1 \cdot C_{33} = -2 \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}$$

**THEOREM 1** The determinant of an $n \times n$ matrix $A$ can be computed by a cofactor expansion across any row or down any column:

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \qquad \text{(expansion across row } i\text{)}$$

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \qquad \text{(expansion down column } j\text{)}$$

Use a matrix of signs to determine $(-1)^{i+j}$

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

**EXAMPLE:** Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$ using cofactor expansion down column 3.

*Solution*

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0 \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 1.$$

**EXAMPLE:** Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{bmatrix}$

*Solution*

$$\begin{array}{c} + \\ - \\ + \\ - \end{array}\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & 1 & 5 \\ 0 & 2 & 1 \\ 0 & 3 & 5 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 & 4 \\ 0 & 2 & 1 \\ 0 & 3 & 5 \end{vmatrix} + 0\begin{vmatrix} 2 & 3 & 4 \\ 2 & 1 & 5 \\ 0 & 3 & 5 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 & 4 \\ 2 & 1 & 5 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= 1\left(2\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} - 0 + 0\right) = 14$$

*Method of cofactor expansion is not practical for large matrices - see Numerical Note on page 190.*

Triangular Matrices:

$$\begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ 0 & 0 & \ddots & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & \ddots & 0 & 0 \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \end{bmatrix}$$

(upper triangular) (lower triangular)

**THEOREM 2**: If $A$ is a triangular matrix, then $\det A$ is the product of the main diagonal entries of $A$.

**EXAMPLE:**

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 4 \end{vmatrix} = \underline{2 \cdot 1 \cdot (-3) \cdot 4} = \underline{\underline{-24}}$$

$$= 2 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & 5 \\ 0 & 0 & 4 \end{vmatrix} = 2 \cdot \left( 1 \cdot \begin{vmatrix} -3 & 5 \\ 0 & 4 \end{vmatrix} \right)$$

$$= 2 \cdot 1 \cdot (-3 \cdot 4) = 2 \cdot 1 \cdot (-3) \cdot 4$$

## 3.2 Properties of Determinants

**THEOREM 3** Let $A$ be a square matrix.

a. If a multiple of one row of $A$ is added to another row of $A$ to produce a matrix $B$, then $\det A = \det B$.

b. If two rows of $A$ are interchanged to produce $B$, then $\det B = -\det A$.

c. If one row of $A$ is multiplied by $k$ to produce $B$, then $\det B = k \cdot \det A$.

**EXAMPLE:** Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix}$.

*Solution*

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix} = 5\begin{vmatrix} 1 & 3 & 4 \\ 2 & 6 & 10 \\ 2 & 7 & 11 \end{vmatrix} = 5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 2 & 7 & 11 \end{vmatrix}$$

$$= 5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = -5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{vmatrix} = \underline{\hspace{3cm}} = \underline{\hspace{1.5cm}}.$$

Theorem 3(c) indicates that $\begin{vmatrix} * & * & * \\ -2k & 5k & 4k \\ * & * & * \end{vmatrix} = k\begin{vmatrix} * & * & * \\ -2 & 5 & 4 \\ * & * & * \end{vmatrix}$.

**EXAMPLE:** Compute $\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix}$

*Solution*

$$\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & -8 & -11 \end{vmatrix}$$

$$= 2(-4)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -8 & -11 \end{vmatrix} = 2(-4)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{vmatrix}$$

$$= 2(-4)(1)(1)(5) = -40$$

**EXAMPLE**: Compute $\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix}$ using a combination of row reduction and cofactor expansion.

*Solution*

$$\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix} = -2\begin{vmatrix} 2 & 3 & 1 \\ 4 & 7 & 3 \\ 1 & 2 & 4 \end{vmatrix} = -2\begin{vmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \\ 0 & 1 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -7 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -7 \\ 0 & 0 & -6 \end{vmatrix} = -2(1)(-1)(-6) = -12.$$

Suspose $A$ has been reduced to $U = \begin{bmatrix} \blacksquare & * & * & \cdots & * \\ 0 & \blacksquare & * & \cdots & * \\ 0 & 0 & \blacksquare & \cdots & * \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix}$ by row replacements and row interchanges, then

$$\det A = \begin{cases} (-1)^r \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

**THEOREM 4** A square matrix is invertible if and only if $\det A \neq 0$.

**THEOREM 5** If $A$ is an $n \times n$ matrix, then $\det A^T = \det A$.

**Partial proof** ($2 \times 2$ case)

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \qquad \text{and}$$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc$$

$$\Rightarrow \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

**THEOREM 6 (Multiplicative Property)**

For $n \times n$ matrices $A$ and $B$, $\det(AB) = (\det A)(\det B)$.

**EXAMPLE:** Compute $\det A^3$ if $\det A = 5$.

*Solution:* $\det A^3 = \det(AAA) = (\det A)(\det A)(\det A)$

$$= \underline{5 \cdot 5 \cdot 5} = \underline{5^3}. = \underline{\underline{125}}$$

**EXAMPLE:** For $n \times n$ matrices $A$ and $B$, show that $A$ is singular if $\det B \neq 0$ and $\det AB = 0$.

*Solution:* *Since*

$$(\det A)(\det B) = \det AB = 0$$

and

$$\det B \neq 0,$$

then $\det A = 0$. Therefore $A$ is singular.

## Inverses: $A$ $n \times n$-matrix

**Recall:**

$A$ has an inverse $A^{-1}$ if there is a matrix $C$ $(= A^{-1})$ such that

$$A \cdot C = C \cdot A = I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$A^{-1}$ exists $\Leftrightarrow$ $\det(A) \neq 0$

If $A^{-1}$ exists, it is unique.

**2x2 case:**

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $2 \times 2$, then:

$|A| = ad - bc \neq 0$: $A^{-1} = \frac{1}{|A|} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$|A| = ad - bc = 0$: $A^{-1}$ does not exist

# Inverses; general case

## Using cofactors:

If $A$ is an $n \times n$-matrix s.t. $\det(A) \neq 0$, then

$$\boxed{A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)}$$

where

$$\operatorname{adj}(A) = (C_{ij})^T = \underbrace{\begin{pmatrix} C_{11} & C_{12} & C_{13} & \cdots & C_{1n} \\ C_{21} & C_{22} & C_{23} & \cdots & C_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & C_{n3} & \cdots & C_{nn} \end{pmatrix}}_{\text{Cofactor matrix}}^T$$

$$= C^T$$

Example:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad C = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

$C_{11} = +|d| = d$

$C_{12} = -c = -c$

$C_{21} = -b$

$C_{22} = a$

$$\operatorname{adj}(A) = C^t = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Ex:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & -2 & 0 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & -2 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 3 \\ -2 & 0 \end{vmatrix} = 1 \cdot (1 \cdot 0 - 3 \cdot (-2)) = \underline{6}$$

$C_{11} = \begin{vmatrix} 2 & 4 \\ -1 & 1 \end{vmatrix} = 6 \qquad C_{21} = -2 \qquad C_{31} = 2$

$C_{12} = -\begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} = 3 \qquad C_{22} = 0 \qquad C_{32} = -3$

$C_{13} = -3 \qquad C_{23} = 2 \qquad C_{33} = 1$

$$\operatorname{adj}(A) = \begin{pmatrix} 6 & -2 & 2 \\ 3 & 0 & -3 \\ -3 & 2 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{6} \cdot \begin{pmatrix} 6 & -2 & 2 \\ 3 & 0 & -3 \\ -3 & 2 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & -1/3 & 1/3 \\ 1/2 & 0 & -1/2 \\ -1/2 & 1/3 & 1/6 \end{pmatrix}}}$$

BI HANDELSHØYSKOLEN
Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo
Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500
www.bi.no
info@bi.no
Org. nr. 971 22 8865

# Inverses using Gauss elimination

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & -1 & 1 \end{pmatrix}$$

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \quad -1 \quad -1$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right) \quad -1 \quad 2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 6 & -3 & 2 & 1 \end{array}\right) : 6$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 2 & -1 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{array}\right) \quad 2 \quad -3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{array}\right) = (I_3 \mid A^{-1})$$

$$A^{-1} = \underline{\underline{\begin{pmatrix} 1 & -1/3 & 1/3 \\ 1/2 & 0 & -1/2 \\ -1/2 & 1/3 & 1/6 \end{pmatrix}}}$$

BI HANDELSHØYSKOLEN
Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo
Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500
www.bi.no
info@bi.no
Org. nr. 971 22 8865

# Solving linear systems using determinants / inverses.

$A \cdot \underline{x} = \underline{b}$

A is a square matrix $(n \times n)$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad\qquad\qquad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$(m = n)$

$\det(A) \neq 0$: $\quad A\underline{x} = \underline{b} \Rightarrow A^{-1}A\underline{x} = A^{-1}\underline{b}$

$\Rightarrow \underline{x} = A^{-1}\underline{b}$

$\det(A) = 0$: no solutions or infinitely many solutions

Example 28. Compute the determinant of $A$ by cofactor expansion along a suitable row where

$$A = \begin{pmatrix} 1 & -1 & -39 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ 2 & 1 & 40 & 2 \end{pmatrix}$$

Solution. By cofactor expansion along the second row we get

$$\begin{aligned} |A| &= 0 \cdot A_{21} + 0 \cdot A_{22} + 1 \cdot A_{23} + 0 \cdot A_{33} \\ &= 1 \cdot (-1)^{2+3} \cdot \begin{vmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 2 \end{vmatrix} \\ &= (-1) \cdot (1 \cdot (-1)^{2+1} \cdot \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} + 0 + 0) \\ &= (-1) \cdot ((-1)((-1) \cdot 2 - 1 \cdot 1) \\ &= (-1) \cdot ((-1)(-3)) \\ &= -3 \end{aligned}$$

Problem 23. Compute the determinant of $A$ by cofactor expansion along a suitable row where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3.9. **Cramer's rule.** Cramer's rule is a useful way to solve a system of linear equations, in particular when we only need to know the value of one of the variables.

Proposition 29. Assume that $A$ is an $n \times n$ matrix and assume that $|A| \neq 0$. Then

$$A\mathbf{x} = \mathbf{b}$$

has a unique solution given as

$$x_i = \frac{|A_{\mathbf{b},i}|}{|A|} \text{ for } i = 1, 2, \ldots, n$$

where $A_{\mathbf{b},i}$ is obtained form by replacing the $i$th column with $\mathbf{b}$.

Let us consider an example.

Example 30. Write the following system of linear equations as $A\mathbf{x} = \mathbf{b}$ and solve it using Cramer's rule:

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - x_2 &= 4. \end{aligned}$$

**Solution.** We get

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

This gives

$$A_{\mathbf{b},1} = \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix} \text{ and } A_{\mathbf{b},2} = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}.$$

Thus we get

$$x_1 = \frac{\begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{(1\cdot(-1) - 4\cdot 1)}{(1\cdot(-1) - 1\cdot 1)} = \frac{-5}{-2} = \frac{5}{2}$$

$$x_2 = \frac{\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{(1\cdot 4 - 1\cdot 1)}{(1\cdot(-1) - 1\cdot 1)} = \frac{3}{-2} = -\frac{3}{2}$$

We try another example.

**Problem 24.** Write the following system of linear equations as $A\mathbf{x} = \mathbf{b}$ and use Cramer's rule to find $x_1$:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_2 - 3x_3 &= 0 \\ x_1 + 4x_2 - 3x_3 &= 0 \end{aligned}$$

3.10. **The Inverse Matrix.** In Lecture 2 we learned about the inverse matrix. Now that we have learned about determinants, we can give a formula for the inverse matrix.

**Proposition 31.** Assume that $A$ is an $n \times n$ matrix. Then $A$ is invertible if and only if $|A| \neq 0$. If $|A| \neq 0$, then

$$A^{-1} = \frac{1}{|A|}\operatorname{adj}(A).$$

We look at some examples.

**Problem 25.** Determine if the following matrix is invertible:

$$\begin{pmatrix} 12 & -3 & 4 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

**Problem 26.** Determine if the given matrix invertible and if so find its inverse.

(a) $A = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ -2 & 1 & 1 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$