Review of Lecture I:

Linear systems:

$$\boxed{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array}}$$

Solutions:

- one solution
- infinitely many solutions
- no solutions

Gauss elimination:

linear system $\rightsquigarrow$

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

augmented matrix

$\Big\downarrow$ elementary row operations

reduced echelon form

$$\left(\begin{array}{cccc|c} \circledast & ? & ? & ? & \cdot \\ 0 & \circledast & ? & ? & \cdot \\ 0 & 0 & 0 & \circledast & \cdot \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Read of the solutions of the linear system. $\longleftrightarrow$

Lecture 2:

Matrix Algebra

Defn: An $m \times n$-matrix is a rectangular array of numbers with m rows and n columns.

Ex: $\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$ is a 2×3-matrix

$$A = \begin{pmatrix} a_{11} & \boxed{a_{12}} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \text{——} \;\; '' \;\; \text{——}$$

Operations on matrices:

A, B $m \times n$-matrices

$$A+B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$$

$$A-B = \quad '' \quad - \quad '' \quad = \begin{pmatrix} a_{11}-b_{11} & -- \\ -- & -- \end{pmatrix}$$

Ex: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1+0 & 2+1 \\ 3-1 & 4+2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ is not defined.

1.3 VECTOR EQUATIONS

Key concepts to master: linear combinations of vectors and a spanning set.

Vector: A matrix with only one column.

Vectors in $\mathbf{R}^n$ (vectors with n entries):

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Geometric Description of $\mathbf{R}^2$

Vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the point (x_1, x_2) in the plane.

$\mathbf{R}^2$ is the set of all points in the plane.

Parallelogram rule for addition of two vectors:

If $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^2$ are represented as points in the plane, then $\mathbf{u}+\mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are $\mathbf{0}$, $\mathbf{u}$ and $\mathbf{v}$. (Note that $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.)

EXAMPLE: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Graphs of $\mathbf{u}, \mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ are given below:

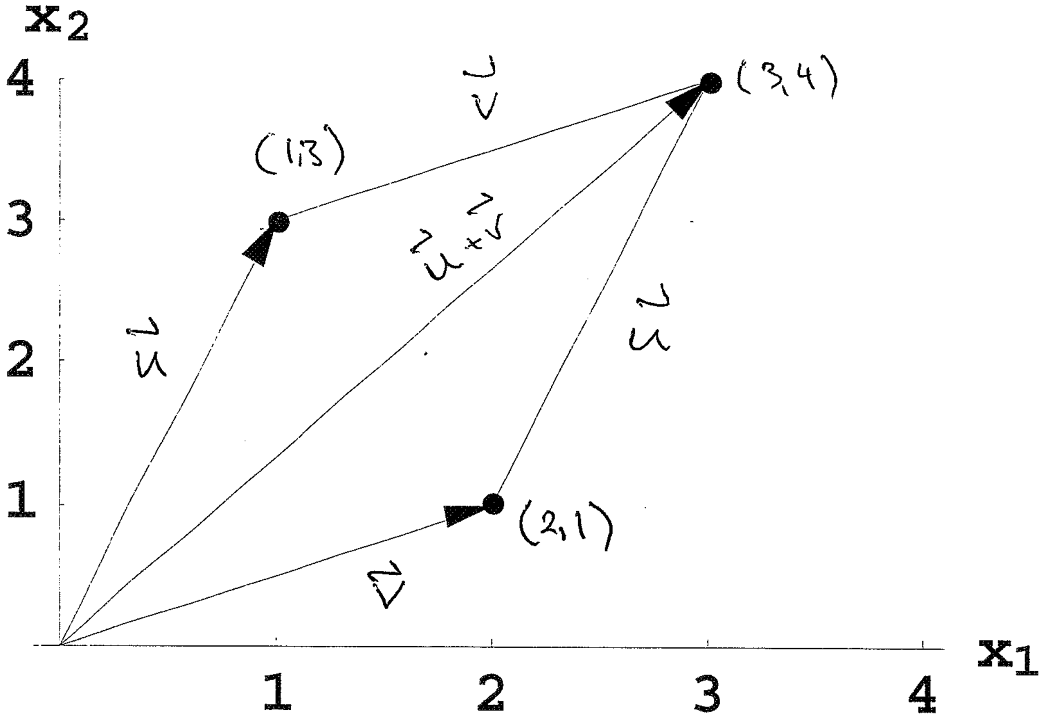
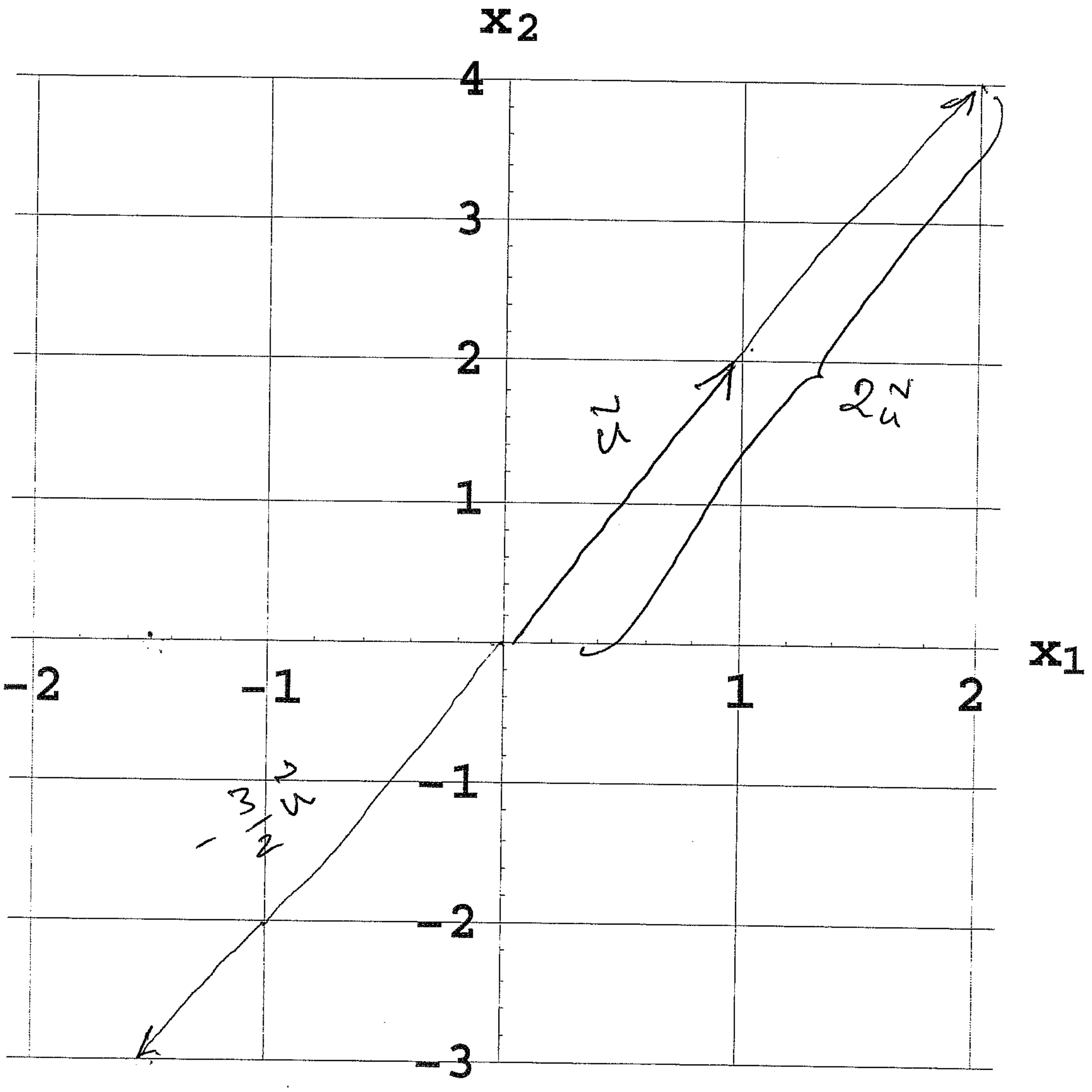


Illustration of the Parallelogram Rule

$$\vec{u}+\vec{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

EXAMPLE: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Express $\mathbf{u}$, $2\mathbf{u}$, and $\frac{-3}{2}\mathbf{u}$ on a graph.



$$2\tilde{u} = 2\cdot\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$-\frac{3}{2}\cdot\tilde{u} = -\frac{3}{2}\cdot\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3/2 \\ -3 \end{pmatrix}$$

Linear Combinations

DEFINITION

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbf{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector **y** defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

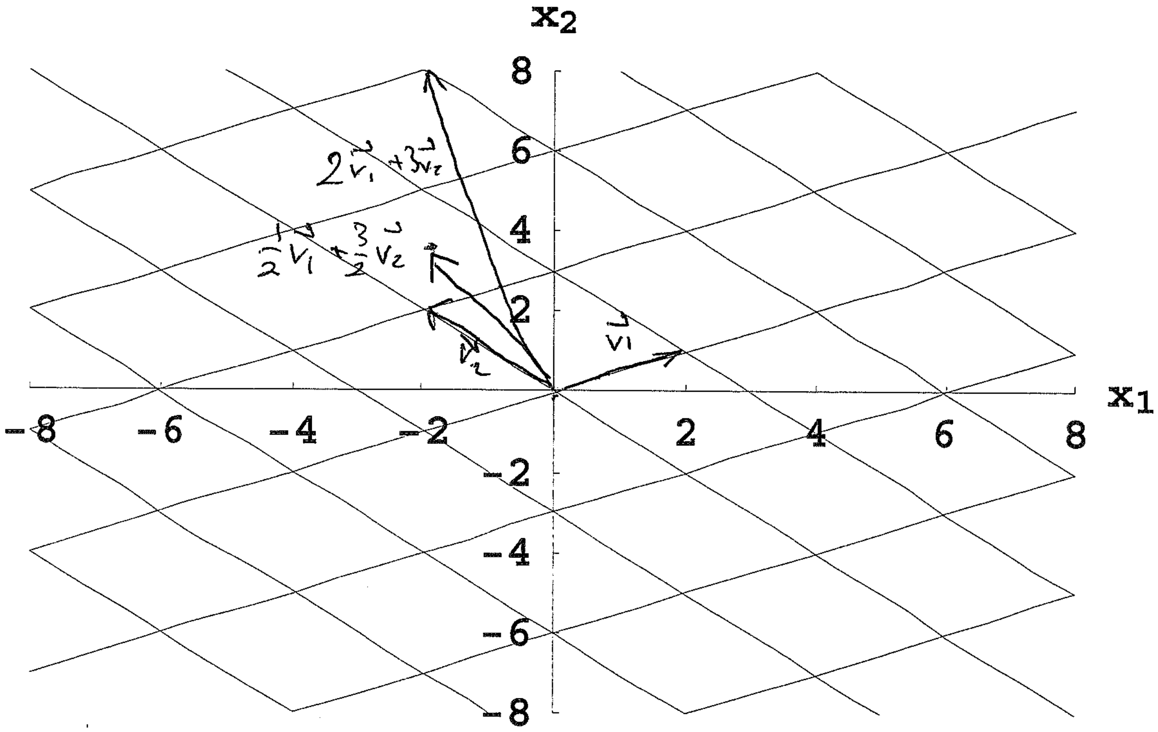
is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ using weights $c_1, c_2, \ldots, c_p$.

Examples of linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$3\mathbf{v}_1 + 2\mathbf{v}_2, \qquad \tfrac{1}{3}\mathbf{v}_1, \qquad \mathbf{v}_1 - 2\mathbf{v}_2, \qquad \mathbf{0}$$

EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Express each of the following as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{a} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$



$2\vec{v}_1 + 3\vec{v}_2$

$\frac{1}{2}v_1 + \frac{3}{2}\vec{v}_2$

EXAMPLE: Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}$,

and $\mathbf{b} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$.

Determine if **b** is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

Solution: Vector **b** is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ if can we find weights x_1, x_2, x_3 such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}.$$

Vector Equation (fill-in):

$$x_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 4 \\ 2 \\ 14 \end{pmatrix} + x_3 \cdot \begin{pmatrix} 3 \\ 6 \\ 10 \end{pmatrix} = \begin{pmatrix} -1 \\ 8 \\ 5 \end{pmatrix}$$

Corresponding System:

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & + & 3x_3 & = & -1 \\ & & 2x_2 & + & 6x_3 & = & 8 \\ 3x_1 & + & 14x_2 & + & 10x_3 & = & -5 \end{array}$$

Corresponding Augmented Matrix:

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{array}\right] \sim \left[\begin{array}{ccc|c} \textcircled{1} & 0 & 0 & 1 \\ 0 & \textcircled{1} & 0 & -2 \\ 0 & 0 & \textcircled{1} & 2 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = \underline{1} \\ x_2 = \underline{-2} \\ x_3 = \underline{2} \end{array}$$

$$1 \cdot \vec{a}_1 + (-2) \cdot \vec{a}_2 + 2 \cdot \vec{a}_3 = \vec{b}$$

Review of the last example: $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{b}$ are columns of the augmented matrix

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{array}\right] \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{array} \end{array}$$

Solution to

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$$

is found by solving the linear system whose augmented matrix is

$$\left[\begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{array}\right].$$

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}.$$

In particular, **b** can be generated by a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ if and only if there is a solution to the linear system corresponding to the augmented matrix.

Ex: Matrix multiplication

$$\begin{pmatrix} 2 & 4 \\ 7 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \cdot \begin{pmatrix} 2 \\ 7 \end{pmatrix} + 2 \cdot \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

$$2 \times \textcircled{2} \text{ —— } \textcircled{2} \times 1$$

$$= \begin{pmatrix} 2 \\ 7 \end{pmatrix} + \begin{pmatrix} 8 \\ -2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 10 \\ 5 \end{pmatrix}}}$$

$$\begin{pmatrix} 2 & 3 & 1 \\ 4 & 0 & 7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad \text{not defined}$$

$$2 \times \textcircled{3} \neq \textcircled{2} \times 1$$

1.4 The Matrix Equation Ax = b

Linear combinations can be viewed as a matrix-vector multiplication.

Definition

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbf{R}^n$, then the **product of A and x**, denoted by $A\mathbf{x}$, is the **linear combination of the columns of A using the corresponding entries in x as weights**. I.e.,

$$A\mathbf{x} = \left[\begin{array}{c|c|c|c} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

EXAMPLE:

$$\begin{bmatrix} 1 & -4 \\ 3 & 2 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ -6 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + -6\begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix} =$$

$$\begin{bmatrix} 7 \\ 21 \\ 0 \end{bmatrix} + \begin{bmatrix} 24 \\ -12 \\ -30 \end{bmatrix} = \begin{bmatrix} 31 \\ 9 \\ -30 \end{bmatrix}$$

EXAMPLE: Write down the system of equations corresponding to the augmented matrix below and then express the system of equations in vector form and finally in the form $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b}$ is a 3×1 vector.

$$\left[\begin{array}{c|c} \underset{\overset{\shortparallel}{A}}{\underbrace{\begin{matrix} 2 & 3 & 4 \\ -3 & 1 & 0 \end{matrix}}} & \underset{\overset{\shortparallel}{\vec{b}}}{\begin{matrix} 9 \\ -2 \end{matrix}} \end{array}\right]$$

Solution: Corresponding system of equations (fill-in)

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 9 \\ -3x_1 + x_2 &= -2 \end{aligned}$$

Vector Equation:

$$x_1\begin{bmatrix} 2 \\ -3 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \end{bmatrix}.$$

Matrix equation (fill-in):

$$\begin{pmatrix} 2 & 3 & 4 \\ -3 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 9 \\ -2 \end{pmatrix}$$

$(A \mid \vec{b})$

$A \cdot \vec{x} = \vec{b}$

$$A \bullet \underset{\overset{\shortparallel}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}}{\underset{\sim}{\vec{x}}} = \underset{\overset{\shortparallel}{\begin{pmatrix} 9 \\ -2 \end{pmatrix}}}{\underset{\sim}{\vec{b}}}$$

Three equivalent ways of viewing a linear system:

1. as a system of linear equations;
2. as a vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$; **or**
3. as a matrix equation $A\mathbf{x} = \mathbf{b}$.

THEOREM 3

If A is a $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbf{R}^m$, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}.$$

Useful Fact:

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a

____________ ____________________ of the columns of A.

EXAMPLE: Let $A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all $\mathbf{b}$?

Solution: Augmented matrix corresponding to $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 4 & 5 & b_1 \\ -3 & -11 & -14 & b_2 \\ 2 & 8 & 10 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & b_1 \\ 0 & 1 & 1 & 3b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 + b_3 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{b}$ is ________ consistent for all $\mathbf{b}$ since some choices of $\mathbf{b}$ make $-2b_1 + b_3$ nonzero.

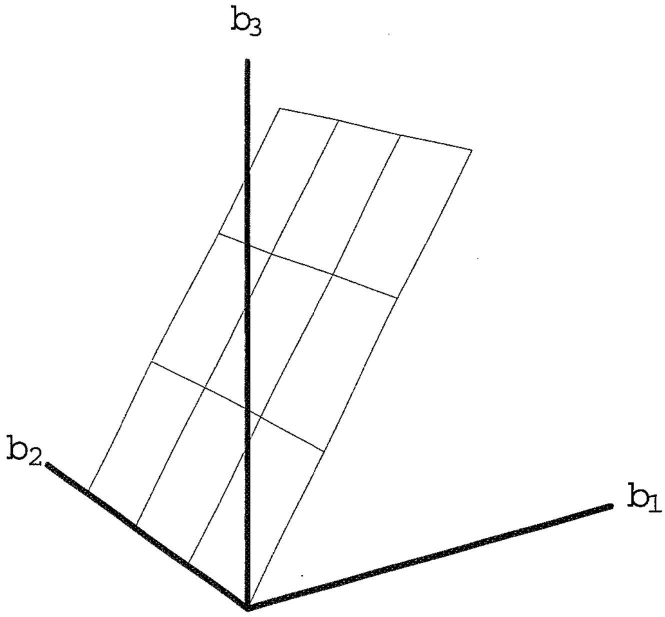
$$A = \begin{array}{c} \left[\begin{array}{ccc} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{array}\right] \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{array} \end{array}$$

The equation $A\mathbf{x} = \mathbf{b}$ is consistent if

$$-2b_1 + b_3 = 0.$$

(equation of a plane in $\mathbf{R}^3$)

$x_1\mathbf{a}_1 + x_2\mathbf{a}_3 + x_3\mathbf{a}_3 = \mathbf{b}$ if and only if $b_3 - 2b_1 = 0$.



Columns of A span a plane
in $\mathbf{R}^3$ through $\mathbf{0}$

Instead, if *any* **b** in $\mathbf{R}^3$ (not just those lying on a particular line or in a plane) can be expressed as a linear combination of the columns of A, then we say that the columns of A span $\mathbf{R}^3$.

Definition

We say that **the columns of** $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_p \end{bmatrix}$ **span** $\mathbf{R}^m$ if ***every*** vector **b** in $\mathbf{R}^m$ is a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_p$

(i.e. $\text{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_p\} = \mathbf{R}^m$).

THEOREM 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent:

a. For each **b** in $\mathbf{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each **b** in $\mathbf{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbf{R}^m$.

d. A has a pivot position in every row.

Proof (outline): Statements (a), (b) and (c) are logically equivalent.

To complete the proof, we need to show that (a) is true when (d) is true and (a) is false when (d) is false.

Suppose (d) is ______________. Then row-reduce the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$:

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} \sim \cdots \sim \begin{bmatrix} U & \mathbf{d} \end{bmatrix}$$

and each row of U has a pivot position and so there is no pivot in the last column of $\begin{bmatrix} U & \mathbf{d} \end{bmatrix}$.

So (a) is ______________.

Now suppose (d) is ______________. Then the last row of $\begin{bmatrix} U & \mathbf{d} \end{bmatrix}$ contains all zeros.

Suppose **d** is a vector with a 1 as the last entry. Then $\begin{bmatrix} U & \mathbf{d} \end{bmatrix}$ represents an inconsistent system.

Row operations are reversible: $\begin{bmatrix} U & \mathbf{d} \end{bmatrix} \sim \cdots \sim \begin{bmatrix} A & \mathbf{b} \end{bmatrix}$

$\Rightarrow \begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ is inconsistent also. So (a) is ______________. ■

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible $\mathbf{b}$?

Solution: A has only _____ columns and therefore has at most _____ pivots.

Since A does not have a pivot in every _______, $A\mathbf{x} = \mathbf{b}$

is ________ ____________________ for all possible $\mathbf{b}$,

according to Theorem 4.

EXAMPLE: Do the columns of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 3 & 9 \end{bmatrix}$ span $\mathbf{R}^3$?

Solution:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 3 & 9 \end{bmatrix} \sim$$

(no pivot in row 2)

By Theorem 4, the columns of A

____.

Another method for computing $A\mathbf{x}$

Read Example 4 on page 44 through Example 5 on page 45 to learn this rule for computing the product $A\mathbf{x}$.

Theorem 5

If A is an $m \times n$ matrix, **u** and **v** are vectors in $\mathbf{R}^n$, and c is a scalar, then:

a. $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

b. $A(c\mathbf{u}) = cA\mathbf{u}$.

2.1 Matrix Operations

Matrix Notation:

Two ways to denote $m \times n$ matrix A:

In terms of the *columns* of A:

$$A = \left[\ \mathbf{a}_1 \mid \mathbf{a}_2 \mid \cdots \mid \mathbf{a}_n\ \right]$$

In terms of the *entries* of A:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Main diagonal entries: $a_{11}, a_{22}, a_{33}, \ldots$

$m = n$
(quadratic matrix)

Zero matrix:

$$0 = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

THEOREM 1

Let A, B, and C be matrices of the same size, and let r and s be scalars. Then

a. $A+B = B+A$

b. $(A+B)+C = A+(B+C)$

c. $A+0 = A$

d. $r(A+B) = rA+rB$

e. $(r+s)A = rA+sA$

f. $r(sA) = (rs)A$

Matrix Multiplication

Multiplying B and $\mathbf{x}$ transforms $\mathbf{x}$ into the vector $B\mathbf{x}$. In turn, if we multiply A and $B\mathbf{x}$, we transform $B\mathbf{x}$ into $A(B\mathbf{x})$. So $A(B\mathbf{x})$ is the composition of two mappings.

Define the product AB so that $A(B\mathbf{x}) = (AB)\mathbf{x}$.

Matrix multiplication – general case

$$\underset{m\times n}{A} = \left(\vec{a}_1 \,\middle|\, \vec{a}_2 \,\middle|\, \cdots \,\middle|\, \vec{a}_n \right)$$

$$\underset{n\times q}{B} = \left(\begin{array}{c|c|c|c} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nq} \end{array} \right)$$

$$\underset{m\times \textcircled{n}}{A}\cdot\underset{\textcircled{n}\times q}{B} = \left(\begin{array}{c|c|c} b_{11}\vec{a}_1 + b_{21}\vec{a}_2 & b_{12}\vec{a}_1 + b_{22}\vec{a}_2 & \\ + \cdots + b_{n1}\vec{a}_n & + \cdots + b_{n2}\vec{a}_n & \cdots \end{array} \right)$$

Ex:

$$\underset{2\times\textcircled{2}}{\overset{\overset{A}{\shortparallel}}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}} \cdot \underset{\textcircled{2}\times 2}{\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}} = \left(2\cdot\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 1\cdot\begin{pmatrix} 2 \\ 4 \end{pmatrix} \,\middle|\, 0\cdot\begin{pmatrix} 1 \\ 3 \end{pmatrix} + (-1)\cdot\begin{pmatrix} 2 \\ 4 \end{pmatrix} \right)$$

$$= \left(\begin{pmatrix} 2 \\ 6 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} \,\middle|\, \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ -4 \end{pmatrix} \right)$$

$$= \left(\begin{array}{c|c} 4 & -2 \\ 10 & -4 \end{array} \right)$$

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\qquad B = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$ $\qquad A \cdot B = ?$

$2 \times \textcircled{2} \qquad \textcircled{2} \times 2 \qquad 2 \times 2$

$$\begin{array}{c|c} & B = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} \\ \hline A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} & AB = \begin{pmatrix} 4 & -2 \\ 10 & -4 \end{pmatrix} \end{array}$$

$$(1 \quad 2) \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} =$$

$$1 \cdot 2 + 2 \cdot 1 = 4$$

$$AB = \begin{pmatrix} 4 & -2 \\ 10 & -4 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 3 & 0 \\ 2 & 7 \\ 1 & 0 \end{pmatrix} \overset{\shortparallel}{B} = \begin{pmatrix} 10 & 14 \\ 14 & 0 \end{pmatrix}$$

$2 \times 3 \qquad 3 \times 2 \qquad 2 \times 2$

$$\begin{array}{c|c} & \begin{pmatrix} 3 & 0 \\ 2 & 7 \\ 1 & 0 \end{pmatrix} \\ \hline \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 2 \end{pmatrix} & \begin{pmatrix} 10 & 14 \\ 14 & 0 \end{pmatrix} \end{array}$$

$$1 \cdot 3 + 2 \cdot 2 + 3 \cdot 1 = 10$$

$B \cdot A =$

$3 \times 2 \qquad 2 \times 3 \qquad 3 \times 3$

Suppose A is $m \times n$ and B is $n \times p$ where

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}.$$

Then

$$B\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_p\mathbf{b}_p$$

and

$$A(B\mathbf{x}) = A(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_p\mathbf{b}_p)$$

$$= A(x_1\mathbf{b}_1) + A(x_2\mathbf{b}_2) + \cdots + A(x_p\mathbf{b}_p)$$

$$= x_1A\mathbf{b}_1 + x_2A\mathbf{b}_2 + \cdots + x_pA\mathbf{b}_p = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}.$$

Therefore,

$$A(B\mathbf{x}) = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}\mathbf{x}.$$

and by defining

$$\boxed{AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}}$$

we have $A(B\mathbf{x}) = (AB)\mathbf{x}$.

EXAMPLE: Compute AB where $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix}$.

Solution:

$$A\mathbf{b}_1 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} -4 \\ -24 \\ 6 \end{bmatrix}, \qquad A\mathbf{b}_2 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -7 \end{bmatrix} = \begin{bmatrix} 2 \\ 26 \\ -7 \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} -4 & 2 \\ -24 & 26 \\ 6 & -7 \end{bmatrix}$$

Note that $A\mathbf{b}_1$ is a linear combination of the columns of A and $A\mathbf{b}_2$ is a linear combination of the columns of A.

> Each column of AB is a linear combination of the columns of A using weights from the corresponding columns of B.

EXAMPLE: If A is 4×3 and B is 3×2, then what are the sizes of AB and BA?

Solution:

$$AB = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} = \begin{bmatrix} \qquad \end{bmatrix}$$

BA would be $\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$

which is ____________________.

$\boxed{\text{If } A \text{ is } m \times n \text{ and } B \text{ is } n \times p, \text{ then } AB \text{ is } m \times p.}$

Row-Column Rule for Computing AB (alternate method)

The definition

$$\boxed{AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}}$$

is good for theoretical work.

When A and B have small sizes, the following method is more efficient when working by hand.

If AB is defined, let $(AB)_{ij}$ denote the entry in the ith row and jth column of AB. Then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

$$\begin{bmatrix} & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{bmatrix} \begin{bmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \\ & b_{nj} & \end{bmatrix}$$

$$= \begin{bmatrix} & & \\ & (AB)_{ij} & \\ & & \end{bmatrix}$$

EXAMPLE $A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix}$. Compute AB, if it is defined.

Solution: Since A is 2×3 and B is 3×2, then AB is defined and AB is ______×______.

$$AB = \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{6} \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{2} & -3 \\ \mathbf{0} & 1 \\ \mathbf{4} & -7 \end{bmatrix} = \begin{bmatrix} \mathbf{28} & \blacksquare \\ \blacksquare & \blacksquare \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{6} \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & \mathbf{-3} \\ 0 & \mathbf{1} \\ 4 & \mathbf{-7} \end{bmatrix} = \begin{bmatrix} 28 & \mathbf{-45} \\ \blacksquare & \blacksquare \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 6 \\ \mathbf{-1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{2} & -3 \\ \mathbf{0} & 1 \\ \mathbf{4} & -7 \end{bmatrix} = \begin{bmatrix} 28 & -45 \\ \mathbf{2} & \blacksquare \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 6 \\ \mathbf{-1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} 2 & \mathbf{-3} \\ 0 & \mathbf{1} \\ 4 & \mathbf{-7} \end{bmatrix} = \begin{bmatrix} 28 & -45 \\ 2 & \mathbf{-4} \end{bmatrix}$$

So $AB = \begin{bmatrix} 28 & -45 \\ 2 & -4 \end{bmatrix}$.

THEOREM 2

Let A be $m \times n$ and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)
b. $A(B+C) = AB + AC$ (left - distributive law)
c. $(B+C)A = BA + CA$ (right-distributive law)
d. $r(AB) = (rA)B = A(rB)$
for any scalar r
e. $I_m A = A = AI_n$ (identity for matrix multiplication)

$$I_m = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & & 1 \end{pmatrix}$$

WARNINGS

Properties above are analogous to properties of real numbers. But **NOT ALL** real number properties correspond to matrix properties.

1. It is not the case that AB always equal BA. (see Example 7, page 114)
2. Even if $AB = AC$, then B may not equal C. (see Exercise 10, page 116)
3. It is possible for $AB = 0$ even if $A \neq 0$ and $B \neq 0$. (see Exercise 12, page 116)

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Powers of A

$$A^k = \underbrace{A\cdots A}_{k}$$

EXAMPLE:

$$\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6 \\ 9 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 21 & 8 \end{bmatrix}$$

$$(A\cdot B)^2 = (A\cdot B)\cdot(A\cdot B) = A\cdot B\cdot A\cdot B \neq A^2\cdot B^2$$

If A is $m \times n$, the **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 8 \\ 7 & 6 & 5 & 4 & 3 \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} 1 & 6 & 7 \\ 2 & 7 & 6 \\ 3 & 8 & 5 \\ 4 & 9 & 4 \\ 5 & 8 & 3 \end{bmatrix}$$

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}$. Compute AB, $(AB)^T$, A^TB^T and B^TA^T.

Solution:

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \qquad\qquad \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} \qquad\qquad \end{bmatrix}$$

$$A^TB^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 10 \\ 2 & 0 & -4 \\ 2 & 1 & 4 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \qquad\qquad \end{bmatrix}$$

THEOREM 3 $A^T = A^t = A'$

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$ (I.e., the transpose of A^T is A)

b. $(A+B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$ (I.e. the transpose of a product of matrices equals the product of their transposes in reverse order.)

EXAMPLE: Prove that $(ABC)^T =$ ________.

Solution: By Theorem 3d,

$$(ABC)^T = ((AB)C)^T = C^T(\qquad)^T$$

$$= C^T(\qquad) = \underline{\qquad\qquad}.$$

2.2 The Inverse of a Matrix

The inverse of a real number a is denoted by a^{-1}. For example, $7^{-1} = 1/7$ and

$$7 \cdot 7^{-1} = 7^{-1} \cdot 7 = 1$$

An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C satisfying

$$CA = AC = I_n$$

where I_n is the $n \times n$ identity matrix. We call C the **inverse** of A .

FACT If A is invertible, then the inverse is unique.

Proof: Assume B and C are both inverses of A. Then

$$B = BI = B(\_\_\_\_\_) = (\_\_\_\_\_\_)\_\_\_\_\_\_ = I\_\_\_\_\_\_ = C.$$

So the inverse is unique since any two inverses coincide.■

The inverse of A is usually denoted by A^{-1}.

We have

$$\boxed{AA^{-1} = A^{-1}A = I_n}$$

Not all $n \times n$ matrices are invertible. A matrix which is *not* invertible is sometimes called a **singular** matrix. An invertible matrix is called **nonsingular** matrix.

Theorem 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

Assume A is any invertible matrix and we wish to solve $A\mathbf{x} = \mathbf{b}$. Then

_____$A\mathbf{x}$ = _____$\mathbf{b}$ and so

$I\mathbf{x}$ = ________ or $\mathbf{x}$ = ________.

Suppose $\mathbf{w}$ is also a solution to $A\mathbf{x} = \mathbf{b}$. Then $A\mathbf{w} = \mathbf{b}$ and

_____$A\mathbf{w}$ = _____$\mathbf{b}$ which means $\mathbf{w} = A^{-1}\mathbf{b}$.

So, $\mathbf{w} = A^{-1}\mathbf{b}$, which is in fact the same solution.

We have proved the following result:

Theorem 5

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbf{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

EXAMPLE: Use the inverse of $A = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$ to solve

$$\begin{array}{rcrcl} -7x_1 & + & 3x_2 & = & 2 \\ 5x_1 & - & 2x_2 & = & 1 \end{array}.$$

Solution: Matrix form of the linear system:

$$\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{14-15} \begin{bmatrix} -2 & -3 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}.$$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

Theorem 6 Suppose A and B are invertible. Then the following results hold:

a. A^{-1} is invertible and $(A^{-1})^{-1} = A$ (i.e. A is the inverse of A^{-1}).

b. AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

c. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Partial proof of part b:

$$(AB)(B^{-1}A^{-1}) = A(\underline{\qquad\qquad\qquad})A^{-1}$$

$$= A(\underline{\qquad\qquad\qquad})A^{-1} = \underline{\qquad\qquad\qquad} = \underline{\qquad\qquad}.$$

Similarly, one can show that $(B^{-1}A^{-1})(AB) = I$.

Theorem 6, part b can be generalized to three or more invertible matrices:

$$(ABC)^{-1} = \underline{\qquad\qquad\qquad}$$

Earlier, we saw a formula for finding the inverse of a 2×2 invertible matrix. How do we find the inverse of an invertible $n \times n$ matrix? To answer this question, we first look at **elementary** matrices.

Ex:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad A^{-1} = ?$$

$$A^{-1} = \frac{1}{1\cdot 4 - 2\cdot 3} \cdot \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}}}$$

Application:

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ 3x_1 + 4x_2 &= -6 \end{aligned} \qquad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$$

$$A \cdot \underline{x} = \underline{b}$$

$$\xleftrightarrow{\quad} \text{ A invertible}$$

$$A^{-1} \cdot A\underline{x} = A^{-1}\underline{b}$$

$$\boxed{\underline{x} = A^{-1} \cdot \underline{b}}$$

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ -6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 4 \\ -6 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 4 \\ -6 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -6 \end{pmatrix} = \begin{pmatrix} -14 \\ 9 \end{pmatrix}$$

$$\underline{\underline{x_1 = -14, \quad x_2 = 9}}$$

Elementary Matrices

Definition

An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.

EXAMPLE: Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

E_1, E_2, and E_3 are elementary matrices. Why?

Observe the following products and describe how these products can be obtained by elementary row operations on A.

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{bmatrix}$$

$$E_2A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$$

$$E_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 3a+g & 3b+h & 3c+i \end{bmatrix}$$

If an elementary row operation is performed on an $m \times n$ *matrix* A*, the resulting matrix can be written as* EA*, where the* $m \times m$ *matrix* E *is created by performing the same row operations on* I_m.

Elementary matrices are *invertible* because row operations are *reversible*. To determine the inverse of an elementary matrix E, determine the elementary row operation needed to transform E back into I and apply this operation to I to find the inverse.

For example,

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad E_3^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Example: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$. Then

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_2(E_1A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$E_3(E_2E_1A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So

$$\boxed{E_3E_2E_1A = I_3}.$$

Then multiplying on the right by A^{-1}, we get

$$E_3E_2E_1A\underline{\hspace{2cm}} = I_3\underline{\hspace{2cm}}.$$

So

$$\boxed{E_3E_2E_1I_3 = A^{-1}}$$

The elementary row operations that row reduce A to $\mathbf{I}_n$ are the same elementary row operations that transform $\mathbf{I}_n$ into $\mathbf{A}^{-1}$.

Theorem 7

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n will also transform I_n to A^{-1}.

Algorithm for finding $\mathbf{A}^{-1}$

Place A and I side-by-side to form an augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$. Then perform row operations on this matrix (which will produce identical operations on A and I). So by Theorem 7:

$$\begin{bmatrix} A & I \end{bmatrix} \text{ will row reduce to } \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

or A is not invertible.

EXAMPLE: Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

Solution:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{bmatrix}$$

So $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$

Order of multiplication is important!

EXAMPLE Suppose A, B, C, and D are invertible $n \times n$ matrices and $A = B(D - I_n)C$.

Solve for D in terms of A, B, C and D.

Solution:

$$\_\_\_\_\_A\_\_\_\_\_ = \_\_\_\_\_B(D - I_n)C\_\_\_\_\_$$

$$D - I_n = B^{-1}AC^{-1}$$

$$D - I_n + \_\_\_\_ = B^{-1}AC^{-1} + \_\_\_\_$$

$$D = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

The quickest way to check if A^{-1} exist, and to find A^{-1}:

$$A = \begin{pmatrix} 4 & 0 \\ 7 & 3 \end{pmatrix}$$

$$(A \,|\, I_2) = \left(\overset{=A}{\boxed{\begin{matrix} 4 & 0 \\ 7 & 3 \end{matrix}}} \;\middle|\; \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right) \begin{matrix} :4 \\ \end{matrix}$$

$$\downarrow$$

$$\left(\begin{matrix} \boxed{1} & 0 \\ 7 & 3 \end{matrix} \;\middle|\; \begin{matrix} 1/4 & 0 \\ 0 & 1 \end{matrix} \right) \begin{matrix} \rfloor -7 \\ \leftarrow \end{matrix}$$

$$\downarrow$$

$$\left(\begin{matrix} \boxed{1} & 0 \\ 0 & \boxed{3} \end{matrix} \;\middle|\; \begin{matrix} 1/4 & 0 \\ -7/4 & 1 \end{matrix} \right) \begin{matrix} \\ :3 \end{matrix}$$

$$\downarrow$$

$$\left(\boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \;\middle|\; \underline{\begin{matrix} 1/4 & 0 \\ -7/12 & 1/3 \end{matrix}} \right) = (I_2 \,|\, A^{-1})$$

Conclusion:

If $(A \,|\, I_n) \rightarrow \cdots \rightarrow (I_n \,|\, B)$,
then A^{-1} exists and $A^{-1} = B$

If not, then A^{-1} does not exist.