Solutions to Lecture 2: Matrix Algebra and Determinants

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Problem 7. Let

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix}.$$

Find $\text{adj}(A)$ and compute $A\,\text{adj}(A)$ and $\text{adj}(A)A$.

Solution.

$$\text{adj}(A) = \begin{pmatrix} 1 & 0 & -2 \\ -2 & -3 & 1 \\ -2 & 0 & 1 \end{pmatrix}$$

$$A\,\text{adj}(A) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ -2 & -3 & 1 \\ -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

$$\text{adj}(A)A = \begin{pmatrix} 1 & 0 & -2 \\ -2 & -3 & 1 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

Problem 9. Compute the determinants

(a) $\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}$ (c) $\begin{vmatrix} a-b & a \\ a & a+b \end{vmatrix}$

Solution.

(a) $\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix} = -14$ (b) $\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = 1$ (c) $\begin{vmatrix} a-b & a \\ a & a+b \end{vmatrix} = -b^2$

Problem 10. Write

$$x_1 + x_2 = 1$$
$$x_1 - 2x_2 = 0$$

as

$$A\mathbf{x} = \mathbf{b}.$$

Find A^{-1} and use this to solve the system of equations.

Solution.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A^{-1} = \frac{1}{(1)(-2)-(1)(1)} \begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A^{-1}\mathbf{b} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

Problem 11. Write the following system of equations as $A\mathbf{x} = \mathbf{b}$:

$$x_1 + 4x_2 + x_3 = 0$$
$$x_1 + 5x_2 + x_3 = 1$$
$$2x_1 + 9x_2 + 3x_3 = 1$$

Find the adjoint matrix $\mathrm{adj}(A)$. Compute $\mathrm{adj}(A)A$ and use this to solve the system of linear equation.

Solution.

$$A = \begin{pmatrix} 1 & 4 & 1 \\ 1 & 5 & 1 \\ 2 & 9 & 3 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathrm{adj}(A) = \begin{pmatrix} 6 & -3 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

$$\mathrm{adj}(A)A = \begin{pmatrix} 6 & -3 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 \\ 1 & 5 & 1 \\ 2 & 9 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 & -3 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$$