FORK1003 - Final test in linear algebra - solutions

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Problem 1. Compute $-2A+5B$ when

$$A = \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}.$$

Solution. $-2A+5B = -2\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix} + 5\begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 9 \\ 14 & 3 \end{pmatrix}$

Problem 2. Compute AB and BA, if possible, for the following:

(1) $A = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 1 & 1 \end{pmatrix}$

(2) $A = \begin{pmatrix} 5 & -3 \\ 10 & 11 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

(3) $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 \\ 3 & 2 \\ -1 & 2 \end{pmatrix}$

Solution. (1)

$$AB = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}\begin{pmatrix} -3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 1 \\ 9 & -3 & -3 \\ -3 & 1 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} -3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} = -5$$

(2)

AB is undefined

BA is undefined

(3)

$$AB = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 3 & 2 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 3 & 2 \\ -1 & 2 \end{pmatrix}$$

BA is not defined

Problem 3. Compute the determinants

(a) $\begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & -13 \\ -2 & 12 \end{vmatrix}$

Solution. (a)

$$\begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix} = 0$$

(b)

$$\begin{vmatrix} 2 & -13 \\ -2 & 12 \end{vmatrix} = -2$$

Problem 4. Write

$$5x_1 - 7x_2 = -2$$
$$7x_1 - 10x_2 = 1$$

as $A\mathbf{x} = \mathbf{b}$. Find A^{-1} and use this to solve the system of equations.

Solution. $A = \begin{pmatrix} 5 & -7 \\ 7 & -10 \end{pmatrix} \implies A^{-1} = \begin{pmatrix} 5 & -7 \\ 7 & -10 \end{pmatrix}^{-1} = \begin{pmatrix} 10 & -7 \\ 7 & -5 \end{pmatrix}$

$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} 10 & -7 \\ 7 & -5 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -27 \\ -19 \end{pmatrix}$

Problem 5. Compute the determinant of A by cofactor expansion along a suitable row and determine if the matrix is invertible.

(a) $A = \begin{pmatrix} 2 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$ (b) $A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 3 & -2 \\ -1 & 0 & 0 & 1 \end{pmatrix}$

Solution. (a)

$\begin{vmatrix} 2 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix} = -2$

(b)

$\begin{vmatrix} 1 & -1 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 3 & -2 \\ -1 & 0 & 0 & 1 \end{vmatrix} = 15$

Problem 6. Write the following system of linear equations as $A\mathbf{x} = \mathbf{b}$ and use Cramers rule to find x_2:

$$2x_1 - x_2 + 2x_3 = 0$$
$$x_1 - 2x_2 - x_3 = 3$$
$$x_1 + x_2 - x_3 = 0$$

Solution. $A = \begin{pmatrix} 2 & -1 & 2 \\ 1 & -2 & -1 \\ 1 & 1 & -1 \end{pmatrix}$, $A_{\mathbf{b},2} = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 3 & -1 \\ 1 & 0 & -1 \end{pmatrix}$

$x_2 = \dfrac{\begin{vmatrix} 2 & 0 & 2 \\ 1 & 3 & -1 \\ 1 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & -1 \\ 1 & 1 & -1 \end{vmatrix}} = \frac{-12}{12} = -1$

Problem 7. Find the inverse of the matrix

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

if it exists.

Solution. $\begin{pmatrix} 2 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ \frac{1}{2} & -1 & -\frac{1}{2} \end{pmatrix}$

Problem 8. Assume that

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 10 & 0 & -1 \end{pmatrix}.$$

Compute A^2. Is A invertible? If so, find the inverse of A without computing cofactors.

Solution. $A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 10 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 10 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies A^{-1} = A$