FORK1003 Linear algebra
Short review of Lecture 2

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Proposition

Let A, B and C be matrices of the same size (order), and let r and s be scalars (numbers). Then:

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A$
4. $r(A + B) = rA + rB$
5. $(r + s)A = rA + sA$
6. $r(sA) = (rs)A$

Proposition

We have the following rules for matrix multiplication:

1. $(AB)C = A(BC)$ (associative law)
2. $A(B + C) = AB + AC$ (left distributive law)
3. $(A + B)C = AC + BC$ (right distributive law)
4. $IA = AI = A$
5. $(AB)^T = B^T A^T$

2.1 Vectors

Definition

A matrix with only one row is called a *row vector* and a matrix with only one column is called a *column vector*.

Example

Show that the system

$$3x_1 + 4x_2 = 5$$
$$7x_1 - 2x_2 = 2$$

of linear equations can be written as $A\mathbf{x} = \mathbf{b}$.

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Solution

We compute

$$A\mathbf{x} = \begin{pmatrix} 3 & 4 \\ 7 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + 4x_2 \\ 7x_1 - 2x_2 \end{pmatrix}$$

and we see that $A\mathbf{x} = \mathbf{b}$ *if and only if* $\begin{pmatrix} 3x_1 + 4x_2 \\ 7x_1 - 2x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

2.2 More on matrix multiplication

Definition

Let A be any matrix. A matrix X is called an inverse of A if

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Definition

If A is a matrix that has an inverse, we write A^{-1} for the inverse of A.

Proposition

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and assume that $ad - bc \neq 0$. Then

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

is the inverse of A.

Example

Find the inverse of $A = \begin{pmatrix} 3 & 4 \\ 7 & -2 \end{pmatrix}$. Solve $\begin{aligned} 3x_1 + 4x_2 &= 5 \\ 7x_1 - 2x_2 &= 2 \end{aligned}$

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Solution

We find that

$$A^{-1} = \frac{1}{3(-2) - 7 \cdot 4} \begin{pmatrix} -2 & -4 \\ -7 & 3 \end{pmatrix}$$

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We have $A\mathbf{x} = \mathbf{b}$ *where* $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ *and* $\mathbf{b} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

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2.3 Determinants

Definition

The determinant of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is written

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

and is defined by

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb.$$

Definition

Let A be an 3×3 matrix. The cofactor A_{ij} is $(-1)^{i+j}$ times the determinant obtained by deleting row i and column j in A.

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Example

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$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix}.$$

Compute the cofactor A_{12}.

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Example

Let

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Compute the cofactor A_{12}.

Solution

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} = (-1)(0 \cdot 1 - 2 \cdot (-1)) = -2$$

Definition

The matrix

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

is called the *cofactor matrix* of A and denoted by $\mathrm{cof}(A)$.

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Definition

The transpose of the cofactor matrix is called the *adjoint* matrix. In symbols

$$\mathrm{adj}(A) = \mathrm{cof}(A)^T.$$