Lecture 3: Determinants and Inverse Matrices

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Contents

Reading. In this lecture we cover topics from Sections 16.2 through 16.6 in [1]. We will only cover parts of Sections 16.3 and 16.4

3.1. **Determinants of Order n and Cofactors.** In the previous lecture we saw how to compute determinants of two by two matrices and how to find cofactors in three by three matrices. Using cofactor expansion we can compute the determinants of three by three matrices.

Definition 1. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

be a 3×3 matrix. Then the determinant of A is given by

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

and this is called the cofactor expansion of $|A|$ along the first row.

We compute an example.

Example 2. Compute the determinant of A by cofactor expansion along the first row where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Solution. We have

$$\begin{aligned} A &= 1 \cdot A_{11} + 2 \cdot A_{12} + 3 \cdot A_{13} \\ &= 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 2 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + 3 \cdot (-1)^{1+3} \cdot \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} \\ &= (2 \cdot 1 - 1 \cdot 1) + 2 \cdot (-1) \cdot (3 \cdot 1 - 1 \cdot 1) + 3 \cdot (3 \cdot 1 - 1 \cdot 2) \\ &= 1 - 4 + 3 \\ &= 0 \end{aligned}$$

The determinant can also be computed by cofactor expansion along another row or along a column. This can reduce the numbers of calculations needed.

Problem 1. Compute the determinant of A by cofactor expansion along a suitable row where

(a) $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix}$

Cofactor expansion can be used to compute determinants of any order. We compute an example with a four by four determinant.

Example 3. Compute the determinant of A by cofactor expansion along a suitable row where

$$A = \begin{pmatrix} 1 & -1 & -39 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 \\ 2 & 1 & 40 & 2 \end{pmatrix}$$

Solution. By cofactor expansion along the second row we get

$$\begin{aligned} |A| &= 0 \cdot A_{21} + 0 \cdot A_{22} + 1 \cdot A_{23} + 0 \cdot A_{33} \\ &= 1 \cdot (-1)^{2+3} \cdot \begin{vmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 2 \end{vmatrix} \\ &= (-1) \cdot (1 \cdot (-1)^{2+1} \cdot \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} + 0 + 0) \\ &= (-1) \cdot ((-1)((-1) \cdot 2 - 1 \cdot 1) \\ &= (-1) \cdot ((-1)(-3)) \\ &= -3 \end{aligned}$$

Problem 2. Compute the determinant of A by cofactor expansion along a suitable row where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3.2. **Cramer's rule.** Cramer's rule is a useful way to solve a system of linear equations, in particular when we only need to know the value of one of the variables.

Proposition 4. Assume that A is an $n \times n$ matrix and assume that $|A| \neq 0$. Then

$$A\mathbf{x} = \mathbf{b}$$

has a unique solution given as

$$x_i = \frac{|A_{\mathbf{b},i}|}{|A|} \text{ for } i = 1, 2, \ldots, n$$

where $A_{\mathbf{b},i}$ is obtained form by replacing the ith column with $\mathbf{b}$.

Let us consider an example.

Example 5. Write the following system of linear equations as $A\mathbf{x} = \mathbf{b}$ and solve it using Cramer's rule:

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - x_2 &= 4. \end{aligned}$$

Solution. We get

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

This gives

$$A_{\mathbf{b},1} = \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix} \text{ and } A_{\mathbf{b},2} = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}.$$

Thus we get

$$x_1 = \frac{\begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{(1 \cdot (-1) - 4 \cdot 1)}{(1 \cdot (-1) - 1 \cdot 1)} = \frac{-5}{-2} = \frac{5}{2}$$

$$x_2 = \frac{\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{(1 \cdot 4 - 1 \cdot 1)}{(1 \cdot (-1) - 1 \cdot 1)} = \frac{3}{-2} = -\frac{3}{2}$$

We try another example.

Problem 3. Write the following system of linear equations as $A\mathbf{x} = \mathbf{b}$ and use Cramer's rule to find x_1:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_2 - 3x_3 &= 0 \\ x_1 + 4x_2 - 3x_3 &= 0 \end{aligned}$$

3.3. **The Inverse Matrix.** In Lecture 2 we learned about the inverse matrix. Now that we have learned about determinants, we can give a formula for the inverse matrix.

Proposition 6. Assume that A is an $n \times n$ matrix. Then A is invertible if and only if $|A| \neq 0$. If $|A| \neq 0$, then

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A).$$

We look at some examples.

Problem 4. Determine if the following matrix is invertible:

$$\begin{pmatrix} 12 & -3 & 4 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Problem 5. Determine if the given matrix invertible and if so find its inverse.

(a) $A = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ -2 & 1 & 1 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

References

1. Knut Sydsæter and Peter J. Hammond, *Essential mathematics for economic analysis*, Prentice Hall, Harlow, 2008.