Lecture 2: Matrix Algebra and Determinants

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Contents

Reading. In this lecture we cover topics from Section 15.4, 15.5, 15.7 and 16.1 in [1].

2.1. **Vectors.** A matrix with only one row is called a *row vector* and a matrix with only one column is called a *column vector*. We refer to both types as vectors. These are typically denoted by small bold letters and not capital letters. If a vector consists of n entries it is called an n-vector.

We may represent 2-vectors in a coordinate system.

Problem 1. Draw the vectors $\mathbf{a} = (1, 4)$ and $\mathbf{b} = (4, 1)$ in a coordinate system. Draw also $\mathbf{a} + \mathbf{b}$.

2.2. **More on Matrix Multiplication.** We will now see how to write a system of linear equations a matrix equation.

Example 1. Show that the system

$$\begin{aligned} 3x_1 + 4x_2 &= 5 \\ 7x_1 - 2x_2 &= 2 \end{aligned}$$

of linear equations can be written as

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{pmatrix} 3 & 4 \\ 7 & -2 \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

Solution. We compute

$$A\mathbf{x} = \begin{pmatrix} 3 & 4 \\ 7 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + 4x_2 \\ 7x_1 - 2x_2 \end{pmatrix}$$

and we see that $A\mathbf{x} = \mathbf{b}$ if and only if $\begin{pmatrix} 3x_1 + 4x_2 \\ 7x_1 - 2x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$. This is the same as saying that $3x_1 + 4x_2 = 5$ and $7x_1 - 2x_2 = 2$.

The advantage of writing a system on matrix form is that this compact form may be used even on very large systems of equations.

Problem 2. Write the following system of equations as $A\mathbf{x} = \mathbf{b}$:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ x_1 + 3x_2 + x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 1 \end{aligned}$$

Matrix notation can also be used to find the solution of a system of linear equations.

Problem 3. Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ 2 & 5 & 3 \end{pmatrix} \text{ and } S = \begin{pmatrix} 4 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}.$$

Compute $(SA)\mathbf{x}$ and use this to solve the system of linear equations in the previous problem.

This suggests the following definition.

Definition 2. Let A be any matrix. A matrix S is called an inverse of A if

$$AS = SA = I.$$

For a 2×2 matrix it is possible to give a formula for the inverse.

Problem 4. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and assume that $ad - bc \neq 0$. Show that

$$\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

is an inverse of A.

An important fact is that the inverse matrix is unique.

Proposition 3. Let A be an $n \times n$ matrix. If A has an inverse, then it is unique.

Proof. Assume that both X and Y are inverses of A. Then

$$Y = IY = (XA)Y = X(AY) = XI = X.$$

□

Since the inverse of a matrix A is unique (if it exists) it is denoted by A^{-1}.

Problem 5. Find the inverse of

$$A = \begin{pmatrix} 3 & 4 \\ 7 & -2 \end{pmatrix}$$

and use this to solve

$$\begin{aligned} 3x_1 + 4x_2 &= 5 \\ 7x_1 - 2x_2 &= 2 \end{aligned}$$

2.3. **Determinants.** We have already encountered determinants of 2×2 matrices.

Definition 4. The determinant of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is written

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

and is defined by

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Consider the following example.

Example 5. Compute the determinants

(a) $\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}$ (c) $\begin{vmatrix} a+b & a-b \\ a-b & a+b \end{vmatrix}$

Solution.

(a) $\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = 1 \cdot 3 - 4 \cdot 2 = -5$

(b) $\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = 2 \cdot 1 - 2 \cdot 3 = -4$

(c) $\begin{vmatrix} a+b & a-b \\ a-b & a+b \end{vmatrix} = (a+b)^2 - (a-b)^2 = 4ab$

We shall later see how to compute the inverse of a 3×3 matrix by computing its so-called *cofactors*.

Definition 6. Let A be an 3×3 matrix. The cofactor A_{ij} is $(-1)^{i+j}$ times the determinant obtained by deleting row i and column j in A.

This definition will be generalized later.

Example 7. Let

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix}.$$

Compute the cofactor A_{12}.

Solution.

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} = (-1)(0 \cdot 1 - 2 \cdot (-1)) = -2$$

Problem 6. Compute some more cofactors of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix}$$

The matrix

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

is called the *cofactor matrix* of A and denoted by $\mathrm{cof}(A)$. If $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix}$ we compute that

$$\mathrm{cof}(A) = \begin{pmatrix} 1 & -2 & -2 \\ 0 & -3 & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

Definition 8. The transpose of the cofactor matrix is called the *adjoint* matrix. In symbols

$$\mathrm{adj}(A) = \mathrm{cof}(A)^T.$$

The adjoint matrix has useful properties.

Problem 7. Let

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix}.$$

Find $\mathrm{adj}(A)$ and compute $A\,\mathrm{adj}(A)$ and $\mathrm{adj}(A)A$.

2.4. **Homework.** You should solve the following problems before the next lecture.

Problem 8. Draw the vectors $\mathbf{a} = (-1, 3)$ and $\mathbf{b} = (4, 2)$ in a coordinate system. Draw also $\mathbf{a} + \mathbf{b}$.

Problem 9. Compute the determinants

(a) $\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}$ (c) $\begin{vmatrix} a-b & a \\ a & a+b \end{vmatrix}$

Problem 10. Write

$$x_1 + x_2 = 1$$
$$x_1 - 2x_2 = 0$$

as

$$A\mathbf{x} = \mathbf{b}.$$

Find A^{-1} and use this to solve the system of equations.

Problem 11. Write the following system of equations as $A\mathbf{x} = \mathbf{b}$:

$$x_1 + 4x_2 + x_3 = 0$$
$$x_1 + 5x_2 + x_3 = 1$$
$$2x_1 + 9x_2 + 3x_3 = 1$$

Find the adjoint matrix $\text{adj}(A)$. Compute $\text{adj}(A)A$ and use this to solve the system of linear equation.

References

1. Knut Sydsæter and Peter J. Hammond, *Essential mathematics for economic analysis*, Prentice Hall, Harlow, 2008.