Lecture 1: Matrices and Matrix Algebra

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Contents

Reading. In this lecture we cover topics from Section 15.2 and 15.3 in [2]. Alternative reading is Section 8.3 in 7th edition of [1] (or Section 9.3 in the 5th or 6th edition).

1.1. **Matrices and Matrix Addition.** Matrices are rectangular arrays of numbers or symbols. Such arrangements of number and symbols occurs in many applications in economics, finance and statistics. Matrices allow for a compact notation and a more efficient handling since it is possible to do arithmetic directly on the matrices.

Definition 1. A matrix is a rectangular array of numbers considered as an entity.

We write down two matrices.

Example 2.

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \\ 6 & 5 \end{pmatrix}$$

Here A is a 2×2 matrix (two by two matrix) and B is a 3×2 matrix. We also say that A has size, order or dimension 2×2.

Definition 3. The sum of two matrices of the same order, is computed by adding the corresponding entries.

It is easy to see how this works in an example.

Example 4.

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 2 \\ 1 & 0 \\ 6 & 5 \end{pmatrix} \qquad C = \begin{pmatrix} 4 & -1 \\ 2 & 0 \end{pmatrix}$$

We can calculate the sum of A and C

$$A + C = \begin{pmatrix} 2 & 1 \\ 3 & 6 \end{pmatrix} + \begin{pmatrix} 4 & -1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 2+4 & 1+(-1) \\ 3+2 & 6+0 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 5 & 6 \end{pmatrix},$$

but the sum of A and B is not defined, since they do not have the same order.

Problem 1. Let

$$A = \begin{pmatrix} -2 & 1 \\ 4 & 2 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ -4 & -2 \\ -1 & 0 \end{pmatrix}.$$

Compute $A + B$.

A matrix can be multiplied by a number.

Definition 5. Let A be a matrix and let k be a real number. Then kA is calculated by multiplying each entry of A by k.

In an example this looks as follows:

Example 6. If

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 6 \end{pmatrix}$$

we get that

$$4A = \begin{pmatrix} 4 \cdot 2 & 4 \cdot 1 \\ 4 \cdot 3 & 4 \cdot 6 \end{pmatrix} = \begin{pmatrix} 8 & 4 \\ 12 & 24 \end{pmatrix}.$$

If A and B are two matrices of the same size we define $A - B$ to be $A + (-1)B$. This means that matrices are subtracted by subtracting the corresponding entries.

Definition 7. We will write 0 for any matrix consisting of only zeros.

Example 8. The following are examples of zero matrices:

$$\underset{2\times 2}{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} \quad \underset{2\times 4}{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}} \quad \underset{3\times 1}{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}$$

Problem 2. Let

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Compute $6A$, $(-1)B$ and $A - B$.

A *square matrix* is a matrix with the same number of rows and columns. In a square matrix, the elements (numbers or symbols) that sit in positions in the matrix where the row number and the column number are the same, constitute the *diagonal.*

Example 9. In the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

the elements 1, 5 and 9, are the elements on the diagonal.

Definition 10. The square matrix of order $n \times n$ that have only 1's on the diagonal and 0's elsewhere, is called the identity matrix and is denoted by I or I_n.

Example 11.

$$I_2 = \underset{2\times 2}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$I_3 = \underset{3\times 3}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

The following is *not* an identity matrix, since it is not a square matrix.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Definition 12. Let A be an $n \times m$ matrix. The transpose of A, denoted A^T, is the $m \times n$ matrix obtained form A by interchanging the rows and columns in A.

For a 2×2 matrix, it looks as follows

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

We look at a 3×3 matrix in an example.

Example 13. Let

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}.$$

Write down A^T.

Solution.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 5 & 1 & 1 \end{pmatrix}.$$

Problem 3. Let

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 2 & -4 & 2 \end{pmatrix}.$$

Compute A^T and I_4^T.

The following rules apply:

Proposition 14. Let A, B and C be matrices of the same size (order), and let r and s be scalars (numbers). Then:

(1) $A + B = B + A$
(2) $(A + B) + C = A + (B + C)$
(3) $A + 0 = A$
(4) $r(A + B) = rA + rB$
(5) $(r + s)A = rA + sA$
(6) $r(sA) = (rs)A$

1.2. **Matrix Multiplication.** Matrix multiplication is slightly more complicated. We start by multiplying 2×2-matrices which are multiplied by the following rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}.$$

Example 15. Compute

$$\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 3 & 1 \end{pmatrix}.$$

Solution. We use the formula given above and obtain

$$\begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 0 + (-2) \cdot 3 & 1 \cdot 1 + (-2) \cdot 1 \\ 0 \cdot 0 + 3 \cdot 3 & 0 \cdot 1 + 3 \cdot 1 \end{pmatrix} = \begin{pmatrix} -6 & -1 \\ 9 & 3 \end{pmatrix}.$$

Note the following pattern:

$$\begin{pmatrix} a & b \\ * & * \end{pmatrix} \begin{pmatrix} e & * \\ g & * \end{pmatrix} = \begin{pmatrix} ae + bg & * \\ * & * \end{pmatrix}$$

As a help, we can use the following diagram

$$\begin{array}{c|c} & \begin{pmatrix} e & * \\ g & * \end{pmatrix} \quad = B \\ \hline A = \begin{pmatrix} a & b \\ * & * \end{pmatrix} & \begin{pmatrix} ae + bg & * \\ * & * \end{pmatrix} \quad = AB \end{array}$$

The same pattern can be use for other matrices, as in the following example:

Example 16. Let

$$A = \begin{pmatrix} 1 & 3 \\ -3 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Compute AB.

Solution.

$$\begin{array}{c|c} & \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad = B \\ \hline A = \begin{pmatrix} 1 & 3 \\ -3 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 \cdot 2 + 3 \cdot 1 & 1 \cdot 0 + 3 \cdot 1 & 1 \cdot 1 + 3 \cdot 0 \\ (-3) \cdot 2 + 0 \cdot 1 & (-3) \cdot 0 + 0 \cdot 1 & (-3) \cdot 1 + 0 \cdot 0 \\ 0 \cdot 2 + 1 \cdot 1 & 0 \cdot 0 + 1 \cdot 1 & 0 \cdot 1 + 1 \cdot 0 \end{pmatrix} \quad = AB \end{array}$$

We get that

$$AB = \begin{pmatrix} 5 & 3 & 1 \\ -6 & 0 & -3 \\ 1 & 1 & 0 \end{pmatrix}.$$

Note the orders of the matrices:

$$\underset{3\times 2}{A}\ \underset{2\times 3}{B} = \underset{3\times 3}{AB}.$$

You should now try to compute some matrices products on your own.

Problem 4. Let

$$A = \begin{pmatrix} 1 & 3 \\ -3 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 3 \\ -3 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Compute AB, BA, AC and BC if possible.

Matrix multiplication follows rules that are similar to the rules for multiplication of numbers.

Proposition 17. We have the following rules for matrix multiplication:

(1) $(AB)C = A(BC)$ (associative law)
(2) $A(B + C) = AB + AC$ (left distributive law)
(3) $(A + B)C = AC + BC$ (right distributive law)
(4) $IA = AI = A$
(5) $(AB)^T = B^T A^T$

Problem 5. Let

$$A = \begin{pmatrix} 1 & 3 \\ -3 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

Verify the proposition for these particular matrices.

1.3. **Homework.** Before you come to the next lecture, you should solve the following problems.

Problem 6. Compute $A+B$ and $5A$ when

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} -2 & -3 \\ -3 & 0 \end{pmatrix}.$$

Problem 7. Compute $A+B$, $A-B$ and $3A-2B$ when

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 3 \\ -3 & 0 \\ 0 & 1 \end{pmatrix}.$$

Problem 8. Compute AB and BA when

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} -2 & -3 \\ -3 & 0 \end{pmatrix}.$$

Problem 9. Compute AB and BA, if possible, for the following:

(1) $A = \begin{pmatrix} 2 & 3 \\ 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(2) $A = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 & 0 \end{pmatrix}$

(3) $A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ 2 & 0 \\ 0 & 1 \end{pmatrix}$

Problem 10. The percentage that will vote for parties Left, Center and Right is given as follows:

	Left	Center	Right	No. of voters
Oslo	46 %	12 %	42 %	550 000
Akershus	40 %	12 %	48 %	500 000
Vestfold	46 %	10 %	44 %	253 000

Use matrix multiplication to compute the total number of voters for each party in the three regions.

References

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2. Knut Sydsæter and Peter J. Hammond, *Essential mathematics for economic analysis*, Prentice Hall, Harlow, 2008.