FORELESNING 5

ELE3717 BI

PLAN:

① Repetisjon: Diskrete stokastiske variable

② Gjennomgang: Oppgaveark 2

③ Kontinuerlige stokastiske variable

Lærebok:

[RJ 2.2, 2.3
(bortsett fra 2.3.3)

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① Repetisjon: Diskrete stokastiske variable

X: diskret stokastisk variabel

$$\begin{cases} f(a) = p(a) := p(X=a) & \text{(sannsynlighetstetthet)} \\ F(b) := p(X \le b) & \text{(kumulativ sannsynlighetsfordeling)} \end{cases}$$

Noen viktige:

a) Uniform (n): $f(i) = \frac{1}{n}$, $i = 1,2,3,\dots,n$

b) Binomisk (n,p): $f(i) = \binom{n}{i} p^i (1-p)^{n-i}$, $i = 0,1,2,\dots,n$

c) Geometrisk (p): $f(i) = (1-p)^{i-1} \cdot p$, $i = 1,2,3,\dots$

d) Poisson (λ): $f(i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$, $i = 0,1,2,3,\dots$

Viktig tilnærming:

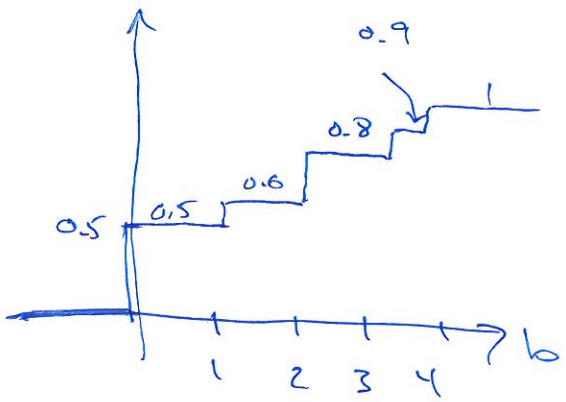
Når n er stor, p er liten og vi setter $\lambda = n \cdot p$, så har vi:

$$\begin{cases} \text{Binom}(n,p) \simeq \text{Poisson}(\lambda) \\ \text{dvs:} \\ \binom{n}{i} p^i (1-p)^{n-i} \simeq e^{-\lambda} \frac{\lambda^i}{i!} \\ \| \qquad\qquad \| \\ p(X=i) \qquad p(Y=i) \\ X \sim \text{Binom}(n,p) \qquad Y \sim \text{Poisson}(\lambda) \end{cases}$$

② Oppgaveark 2: 9, 14, 16, 30, 32

2.9

$$F(b) = \begin{cases} 0, & b \le 0 \\ 0.5, & 0 < b \le 1 \\ 0.6, & 1 \le b \le 2 \\ 0.8, & 2 < b \le 3 \\ 0.9, & 3 < b \le 3.5 \\ 1, & 3.5 < b \end{cases}$$



$$f(a) = \begin{cases} 0.5, & a = 0 \\ 0.1, & a = 1 \\ 0.2, & a = 2 \\ 0.1, & a = 3 \\ 0.1, & a = 3.5 \end{cases}$$

2.14 $X \sim \text{Binom}(n=6,\ p=1/2)$

$$p(X=i) = \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i} = \binom{6}{i} \cdot \left(\frac{1}{2}\right)^i \cdot \left(\frac{1}{2}\right)^{6-i} = \binom{6}{i} \cdot \left(\frac{1}{2}\right)^6$$

$\binom{6}{i}$:

0	1	2	3	4	5	6
1	6	15	20	15	6	1

Maks for $i=3$

$$p(X=3) = 20 \cdot \left(\frac{1}{2}\right)^6$$

2.16: X = antall passasjerer som møter

$\simeq \text{Binom}(n=52,\ p=0.95)$

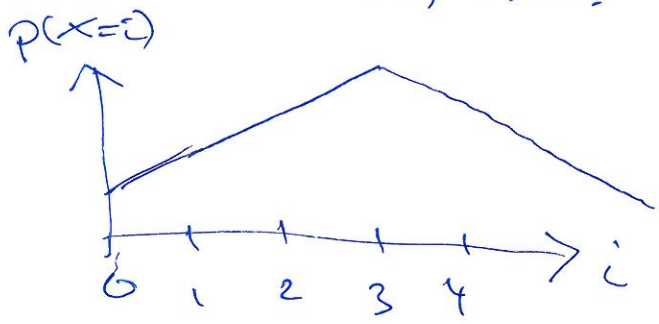
$$p(X \le 50) = 1 - p(X=51) - p(X=52)$$

$$= 1 - \binom{52}{51} \cdot 0.95^{51} \cdot 0.05^1 - \binom{52}{52} \cdot 0.95^{52} \cdot 0.05^0$$

$$= 1 - 52 \cdot 0.95^{51} \cdot 0.05 - 0.95^{52} \simeq 0.74$$

2.30. $X \sim \text{Poisson}(\lambda)$ $\lambda > 0$

$P(X=i)$, $i = 0, 1, 2, 3, \ldots$



$$P(X=i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$
$$P(X=i-1) = e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!}$$

$$\frac{P(X=i)}{p(X=i-1)} = \frac{e^{-\lambda} \cdot \frac{\lambda^i}{i!}}{e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!}} = \frac{\lambda^i}{i!} \cdot \frac{(i-1)!}{\lambda^{i-1}} = \frac{\lambda}{i}$$

$$p(X=i) = \left(\frac{\lambda}{i}\right) \cdot P(X=i-1)$$

$i \leq \lambda$: $p(X=i) \geq p(X=i-1)$ økende fn.

$i \geq \lambda$: $p(X=i) \leq p(X=i-1)$ avtagende fn.

Maks for i = største heltall $\leq \lambda$.

2.32: X = antall gevinster

$X \sim$ Binom $(n=50,\ p=\frac{1}{100})$

$\approx$ Poisson $(\lambda = np = 50 \cdot \frac{1}{100} = \underline{1/2})$

a) $P(X \geq 1) = 1 - P(X=0) \approx 1 - e^{-1/2} \cdot \frac{1}{1}$

$= 1 - e^{-1/2} \approx \underline{0.393}$

b) $P(X=1) \approx e^{-1/2} \cdot \frac{\lambda^1}{1!} = e^{-1/2} \cdot \frac{1}{2} \approx \underline{0.303}$

c) $P(X \geq 2) = P(X \geq 1) - P(X=1) \approx \underline{0.090}$

$= 1 - P(X=0) - P(X=1)$

$= 1 - e^{-1/2} - \frac{1}{2}e^{-1/2} = 1 - \frac{3}{2}e^{-1/2} \approx \underline{0.90}$

2.4-2.5:

a) $X = \max\{X_1, X_2\}$

X_1 \ X_2	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

2.4: Mulige verdier for X:

1, 2, 3, 4, 5, 6

2.5:

$P(X=1) = 1/36$

$P(X=2) = 3/36$

$P(X=3) = 5/36$

$P(X=4) = 7/36$

$P(X=5) = 9/36$

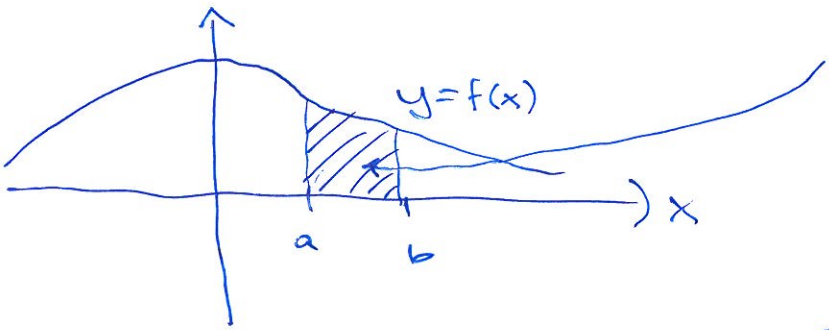
$P(X=6) = 11/36$

(3) Kontinuerlige stokastiske variable

Stokastisk variabel der de mulige verdiene er kontinuerlig (dvs inneholder et intervall $[a,b] = \{x: a \le x \le b\}$).

Sannsynlighetstettheten til X er en funksjon $f(x) = f_X(x)$ slik at

$$p(a \le X \le b) = \int_a^b f(x)\,dx \quad \text{for alle } a \le b$$



$p(a \le X \le b)$ = skravert areal

Krav til $f(x)$:

i) $f(x) \ge 0$ for alle x

ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$F(b) = p(X \le b) = \int_{-\infty}^{b} f(x)\,dx$$

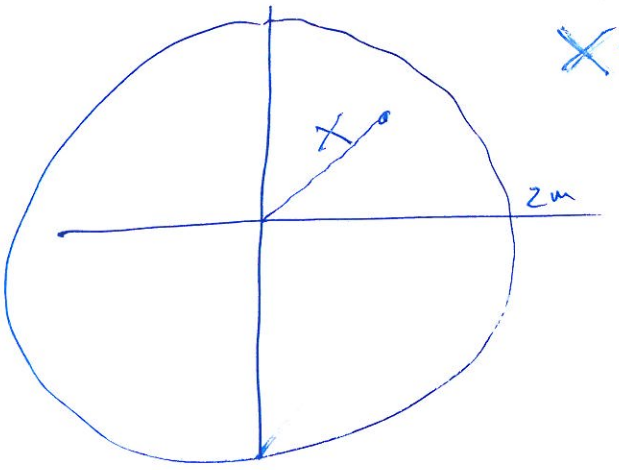
$$\frac{d}{db} F(b) = f(b) \quad \text{dvs} \quad F'(b) = f(b)$$

Merk: $p(X = b) = 0$!

(punktsannsynligheten til en kontinuerlig stok. variabel er null)

Eks: Vi kaster pil på en blink.

~~Anta at~~



X = avstand fra treffpunkt til senter av blinken

Anta:

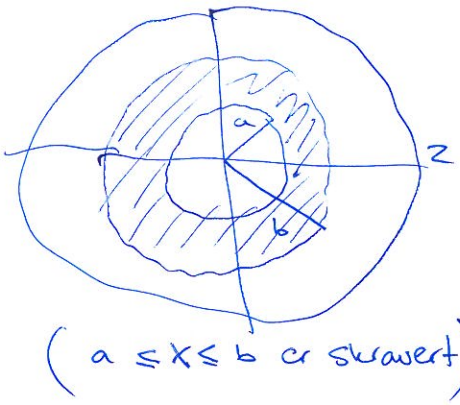
i) Alle treffer innenfor radius 2m

ii) Sannsynligheten for å treffe innenfor et område A er proposjonalt med arealet.

Mulige verdier: $[0,2]$

Hva er $f(x)$?

Defn: $p(a \le X \le b) = \int_a^b f(x)\,dx$



($a \le X \le b$ er skravert)

$$p(a \le X \le b) = \frac{\text{skravert areal}}{\text{hele arealet}} = \frac{\pi b^2 - \pi a^2}{\pi \cdot 2^2} = \frac{b^2 - a^2}{4}$$

$$p(0.1 \le X \le 0.2) = \frac{0.2^2 - 0.1^2}{4} = \frac{0.03}{4} = 0.0075$$

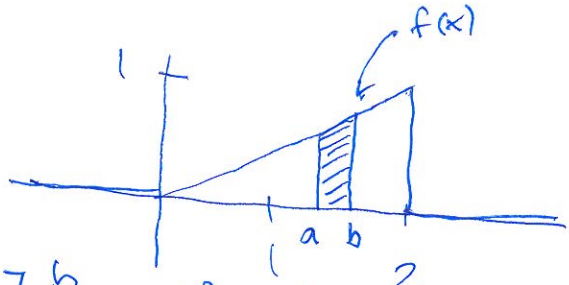
$$p(1.1 \le X \le 1.2) = \frac{1.2^2 - 1.1^2}{4} = \frac{0.23}{4} = 0.0575$$

$$F(b) = p(X \le b) = p(0 \le X \le b) = \frac{b^2 - 0^2}{4} = \frac{b^2}{4}$$

$$f(b) = F'(b) = \frac{2b}{4} = \frac{b}{2}$$

Konklusjon:

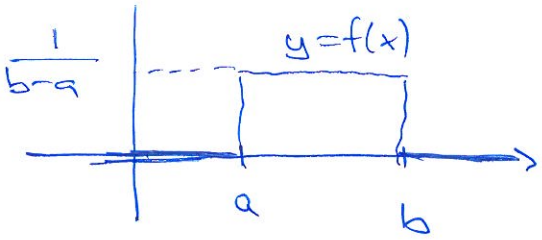
$$f(x) = \frac{x}{2}, \quad 0 \le x \le 2$$



$$p(a \le X \le b) = \int_a^b \frac{x}{2}\,dx = \left[\frac{x^2}{4}\right]_a^b = \frac{b^2}{4} - \frac{a^2}{4} \quad \text{ok.}$$

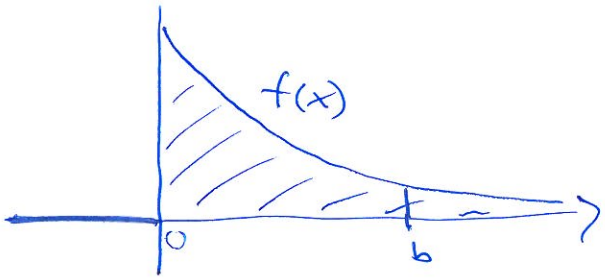
Viktige kontinuerlige fordelinger:

a) Uniform på $[a,b]$: Uniform($[a,b]$)



$$f(x) = \begin{cases} \frac{1}{b-a}, & x \in [a,b] \\ 0, & \text{ellers} \end{cases}$$

b) Eksponensial: $(\lambda > 0)$ Exp(λ)



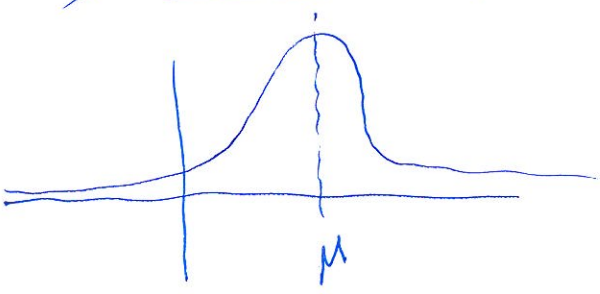
$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$F(b) = \int_{-\infty}^{b} f(x)\,dx = \int_{0}^{b} \lambda e^{-\lambda x}\,dx = \left[-e^{-\lambda x}\right]_0^b$$

$$= -e^{-\lambda b} + e^0 = 1 - e^{-\lambda b}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{b\to\infty} F(b) = \lim_{b\to\infty}\left(1 - e^{-\lambda b}\right) = \underline{\underline{1}} \quad \text{ok.}$$

c) Normal: μ, σ Norm(μ,σ)



$$f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for alle } x$$

Vi fortsetter ved normalfordelingen neste gang.