Løsn: Oppgaveark 19

Oppgave 1

Finn den generelle løsningen av $y' + \frac{1}{2}y = \frac{1}{4}$. Er løsningen stabil? Bestem likevektstilstanden. Tegn noen typiske løsninger i et koordinatsystem.

$y' + \frac{1}{2}y = \frac{1}{4}$

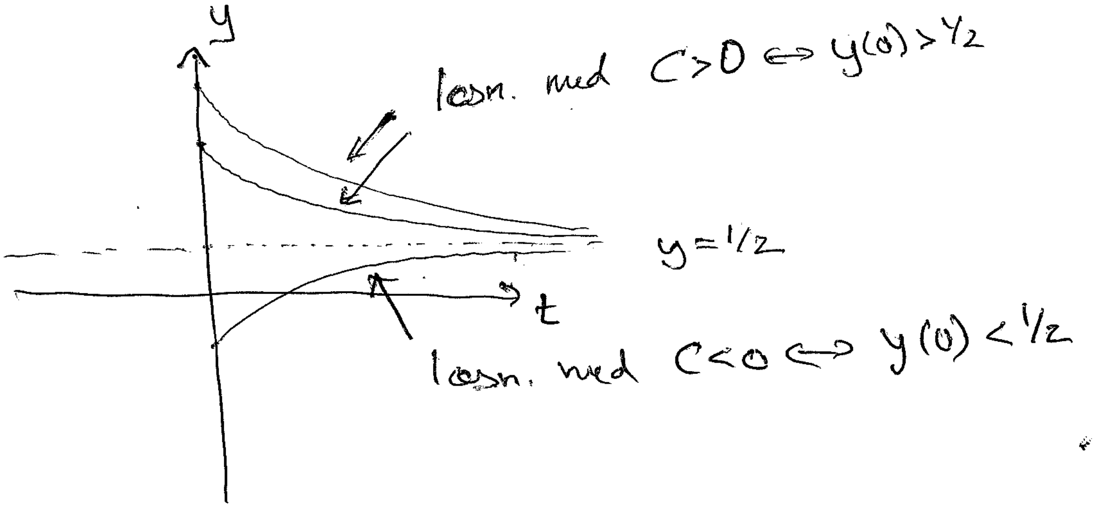
$a = 1/2,\ b = 1/4$

$y = \frac{b}{a} + Ce^{-at}$

$= \frac{1/4}{1/2} + Ce^{-\frac{1}{2}t} = \frac{1}{2} + Ce^{-\frac{1}{2}t}$

$\lim_{t\to\infty} y(t) = \frac{1}{2}$ stabil

likevekt: $y = \frac{1}{2}$



$y(0) = \frac{1}{2} + C$

OPPGAVE 2

Finn den generelle løsningen i hvert tilfelle:

a) $y' + y = 10$

b) $y' - 3y = 27$

c) $4y' + 5y = 100$

Bruker:

$y' + ay = b$

$\Rightarrow y = \frac{b}{a} + C \cdot e^{-at}$

a) $y' + y = 10$

$y = \frac{10}{1} + C \cdot e^{-1 \cdot t} = \underline{\underline{10 + Ce^{-t}}}$

b) $y' - 3y = 27$

$y = \frac{27}{-3} + C \cdot e^{+3t} = \underline{\underline{-9 + Ce^{3t}}}$

c) $4y' + 5y = 100$

$y' + \frac{5}{4}y = 25$

$y = \frac{25}{5/4} + C \cdot e^{-5/4\,t} = \underline{\underline{20 + C \cdot e^{-\frac{5}{4}t}}}$

c) Alt. for generell løsn. i 2c): $y = y_h + y_p = \underline{\underline{Ce^{-2t} + \frac{1}{2}t^2 - \frac{1}{2}t + \frac{1}{4}}}$

y_h: $y' + 2y = 0$

$y_h = 0 + Ce^{-2t} = \underline{Ce^{-2t}}$

y_p: Prøver $y = At^2 + Bt + C \Rightarrow$ VS: $y' + 2y = (2At + B) + 2(At^2 + Bt + C) = (2A)t^2 + (2A + 2B)t + (B + 2C)$

HS: $t^2 = 1 \cdot t^2 + 0 \cdot t + 0$

$\Downarrow$

$2A = 1$ $\quad A = 1/2$

$2A + 2B = 0$ $\quad B = -1/2$

$B + 2C = 0$ $\quad C = 1/4$

$\Downarrow$

$y_p = \underline{\frac{1}{2}t^2 - \frac{1}{2}t + \frac{1}{4}}$

OPPGAVE 3

Finn i hvert enkelt tilfelle den generelle løsningen. Finn også den partikulære løsningen som tilfredsstiller $y(0) = 1$.

a) $y' - 3y = 5$

b) $3y' + 2y + 16 = 0$

c) $y' + 2y = t^2$

a) $y' - 3y = 5 \Rightarrow y = \frac{5}{-3} + C \cdot e^{3t} = \underline{\underline{-\frac{5}{3} + Ce^{3t}}}$

$\underline{y(0) = 1}$:

$1 = -\frac{5}{3} + C \cdot e^0 \Rightarrow C = 1 + \frac{5}{3} = \frac{8}{3}$

$\underline{\underline{y = -\frac{5}{3} + \frac{8}{3}e^{3t}}}$

b) $3y' + 2y + 16 = 0$

$3y' + 2y = -16$

$y' + \frac{2}{3}y = -\frac{16}{3} \Rightarrow y = \frac{-16/3}{2/3} + C \cdot e^{-\frac{2}{3}t} = \underline{\underline{-8 + Ce^{-\frac{2}{3}t}}}$

$\underline{y(0) = 1}$: $1 = -8 + C \cdot e^0 \Rightarrow C = 1 + 8 = 9$

$y = \underline{\underline{-8 + 9e^{-\frac{2}{3}t}}}$

c) $y' + 2y = t^2$ Int. faktor: e^{2t}

$(y \cdot e^{2t})' = t^2 e^{2t}$

$ye^{2t} = \int t^2 e^{2t}\,dt = t^2 \cdot \frac{1}{2}e^{2t} - \int 2t \cdot \frac{1}{2}e^{2t}\,dt$

$= \frac{1}{2}t^2 e^{2t} - \int t e^{2t}\,dt = \frac{1}{2}t^2 e^{2t} - \left(t \cdot \frac{1}{2}e^{2t} - \int 1 \cdot \frac{1}{2}e^{2t}\,dt\right)$

$ye^{2t} = \frac{1}{2}t^2 e^{2t} - \frac{1}{2}te^{2t} + \frac{1}{4}e^{2t} + C$

$y = \underline{\underline{\frac{1}{2}t^2 - \frac{1}{2}t + \frac{1}{4} + Ce^{-2t}}}$

$y(0) = 1$: $1 = \frac{1}{4} + C \cdot e^0 \Rightarrow C = 3/4 \Rightarrow y = \underline{\frac{1}{2}t^2 - \frac{1}{2}t + \frac{1}{4} + \frac{3}{4}e^{-2t}}$

Oppgave 4

Finn den generelle løsningen:

a) $ty' + 2y + t = 0, \quad t \neq 0$

b) $y' - \frac{1}{4}y = t, \quad t > 0$

c) $y' - \frac{t}{t^2-1}\, y = t, \quad t > 1$

a) $ty' + 2y + t = 0 \quad , \quad t \neq 0$

$y' + \frac{2}{t}y = -1$ Int. faktor: $u = e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2$

$(y \cdot t^2)' = -t^2$

$yt^2 = \int -t^2\,dt = -\frac{1}{3}t^3 + C$

$\underline{\underline{y = -\frac{1}{3}t + \frac{C}{t^2}}}$

b) $y' - \frac{1}{4}y = t \quad , \quad t > 0$ Int. faktor: $u = e^{-1/4t}$

$(y \cdot e^{-1/4t})' = te^{-1/4t}$

$y \cdot e^{-1/4t} = \int \underset{v}{t}\,\underset{u'}{e^{-1/4t}}\,dt = -4te^{-1/4t} - \int -4e^{-1/4t} \cdot 1\,dt$

$\begin{pmatrix} u = -4e^{-1/4t} \\ v' = 1 \end{pmatrix} \quad = -4te^{-1/4t} + 4 \cdot (-4)e^{-1/4t} + C$

$\underline{\underline{y = -4t - 16 + Ce^{1/4t}}}$

c) $y' - \frac{t}{t^2-1}y = t \quad , \quad t > 1$

$\int -\frac{t}{t^2-1}\,dt = -\frac{1}{2}\ln(t^2-1) + C$

$u = e^{-\frac{1}{2}\ln(t^2-1)} = (t^2-1)^{-1/2} = \frac{1}{\sqrt{t^2-1}}$

$(y \cdot (t^2-1)^{-1/2})' = t \cdot (t^2-1)^{-1/2}$

$y \cdot (t^2-1)^{-1/2} = \int t \cdot (t^2-1)^{-1/2}\,dt = (t^2-1)^{1/2} + C$

$y = (t^2-1) + C(t^2-1)^{1/2} = \underline{\underline{t^2 - 1 + C \cdot \sqrt{t^2-1}}}$

Oppgave 5

Finn i hvert enkelt tilfelle den generelle løsningen. Finn også den partikulære løsningen som tilfredsstiller den gitte initialbetingelsen.

a) $y' = 4(y-1)(y-3), \quad y(0) = 2$

b) $e^{2t}y' - y^2 - 2y = 1, \quad y(1) = 1$

c) $y' - \frac{t}{t^2-1}y = 0, \quad y(0) = 1$

$\frac{1}{(y-3)(y-1)} = \frac{A}{y-1} + \frac{B}{y-3}$

$1 = A(y-3) + B(y-1)$

$y=3: \quad 1 = 2B \Rightarrow B = \frac{1}{2}$

$y=1: \quad 1 = -2A \Rightarrow A = -\frac{1}{2}$

$\frac{1}{(y-1)(y-3)} = \frac{-1/2}{y-1} + \frac{1/2}{y-3}$

a) $y' = 4(y-1)(y-3), \quad y(0) = 2$ — separabel, ikke lineær

$\frac{1}{(y-1)(y-3)}y' = 4$

$\int \frac{1}{(y-1)(y-3)}\,dy = \int 4\,dt = 4t + C$

$\int \frac{-1/2}{y-1} + \frac{1/2}{y-3}\,dy = -\frac{1}{2}\ln|y-1| + \frac{1}{2}\ln|y-3| = 4t + C$

$\frac{1}{2}\ln\left|\frac{y-3}{y-1}\right| = 4t + C \qquad \ln\left|\frac{y-3}{y-1}\right| = 8t + 2C$

$\frac{y-3}{y-1} = \pm e^{8t}\cdot e^{2C} = K\cdot e^{8t}$

$y - 3 = Ke^{8t}(y-1)$

$y(1 - Ke^{8t}) = 3 - Ke^{8t}$

$y = \frac{3 - Ke^{8t}}{1 - Ke^{8t}}$

$y(0) = 2: \quad t = 0, y = 2$

$\frac{2-3}{2-1} = K\cdot e^0 = K \Rightarrow K = -1$

$y = \frac{3 + e^{8t}}{1 + e^{8t}}$

b) $e^{2t}y' - y^2 - 2y = 1$ — separabel, ikke lineær

$e^{2t}y' = y^2 + 2y + 1 = (y+1)^2$

$\frac{1}{(y+1)^2}y' = \frac{1}{e^{2t}} = e^{-2t}$

$\int \frac{1}{(y+1)^2}\,dy = \int e^{-2t}\,dt$

$-\frac{1}{(y+1)} = -\frac{1}{2}e^{-2t} + C$

$\frac{1}{y+1} = \frac{1}{2}e^{-2t} - C = \frac{e^{-2t} - 2C}{2}$

$y + 1 = \frac{2}{e^{-2t} - 2C}$

$y = \frac{2}{e^{-2t} - 2C} - 1$

$y(1) = 1: \quad t = 1, y = 1$

$-\frac{1}{2} = -\frac{1}{2}e^{-2} + C$

$C = \frac{1}{2}e^{-2} - \frac{1}{2}$

$= \frac{1}{2}(e^{-2} - 1)$

$y = \frac{2}{e^{-2t} - e^{-2} + 1} - 1$

c) $y' - \frac{t}{t^2-1} y = 0$, lineær og separabel
løser som lineær, bruker 4c):

$u = |t^2-1|^{-1/2}$

Må bruke absoluttverdi, fordi $(t^2-1) > 0$ holder ikke nødvendigvis men for $t>1$ i 4c) gjorde det

$$\left(y \cdot |t^2-1|^{-1/2}\right)' = 0 \cdot |t^2-1|^{-1/2} = 0$$

$$y \cdot |t^2-1|^{-1/2} = \int 0 \, dt = C$$

$$y = C \cdot |t^2-1|^{1/2} = \underline{\underline{C \cdot \sqrt{|t^2-1|}}}$$

$\underline{y(0) = 1:}$

$t=0,\ y=1 \Rightarrow 1 = C \cdot \sqrt{|-1|} = C \cdot \sqrt{1} = C \Rightarrow \underline{C=1}$

$y = \sqrt{|t^2-1|} = \underline{\underline{\sqrt{1-t^2}}}$ for $-1 < t < 1$
(i nærheten av $t=0$)