Eks: N innbyggere $\begin{cases} Np & \text{for EU} \\ N(1-p) & \text{mot EU} \end{cases}$

p = andelen for EU (ukjent)

Utvalg: n personer

$$X_i = \begin{cases} 1, & \text{person } i \text{ for EU} \\ 0, & \text{— " — mot EU} \end{cases}$$

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{X_1 + X_2 + \dots + X_n}{n}$$

(a) $E[\bar{X}] = \sum_{i=1}^{n} \frac{1}{n} E[X_i] = \frac{n \cdot p}{n} = p$,

siden $E[X_i] = \frac{Np}{N} = p$ og $\sum_{i=1}^{n} E(X_i) = np$.

(b) $\mathrm{Var}(\bar{X}) = \mathrm{Var}\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n^2} \cdot \mathrm{Var}(X_1 + \dots + X_n)$

$= \frac{1}{n^2} \cdot \mathrm{Cov}(X_1 + X_2 + \dots + X_n, X_1 + X_2 + \dots + X_n)$

$= \frac{1}{n^2} \Big(\mathrm{Cov}(X_1, X_1) + \mathrm{Cov}(X_2, X_2) + \dots + \mathrm{Cov}(X_n, X_n)$
$+ \mathrm{Cov}(X_1, X_2) + \dots + \mathrm{Cov}(X_n, X_{n-1}) \Big)$

$= \frac{1}{n^2} \Big(n \cdot \mathrm{Cov}(X_i, X_i) + (n^2 - n)\,\mathrm{Cov}(X_i, X_j) \Big)$

$= \frac{1}{n^2} \left(n \cdot p(1-p) + n(n-1) \cdot \left[\frac{Np-1}{N-1} \cdot \frac{Np}{N} - p^2 \right] \right)$

$= \frac{p(1-p)}{n} + \frac{n-1}{n} \cdot \frac{(Np-1)p - (N-1)p^2}{N-1}$

$= \frac{p(1-p)}{n} + \frac{(n-1)(p^2-p)}{n(N-1)} = \frac{p(1-p)}{n}\left(1 + \frac{n-1}{N-1}\right)$

$= \frac{p(1-p)}{n} \cdot \frac{(N-1)-(n-1)}{(N-1)} = \frac{p(1-p)(N-n)}{n \cdot (N-1)}$,

Siden $E[X_i^2] = E[X_i] = p \Rightarrow \mathrm{Var}(X_i) = p - p^2 = p(1-p)$

$E[X_i X_j] = P(X_i = 1 \mid X_j = 1) \cdot P(X_j = 1)$

$= \frac{Np-1}{N-1} \cdot \frac{Np}{N} \Rightarrow \mathrm{Cov}(X_i, X_j) = \frac{(Np-1)p}{N-1} - p^2 \quad i \neq j$