

## Exercise session problems

### Problem 1.

Consider the vectors given by

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

Draw these vectors in a two-dimensional coordinate system. Then compute the following vectors, and draw them into the same coordinate system:

a)  $\mathbf{u} + \mathbf{v}$       b)  $\mathbf{v} + \mathbf{w}$       c)  $\mathbf{v} - \mathbf{w}$       d)  $2\mathbf{u}$       e)  $-\mathbf{v}$       f)  $3\mathbf{u} + \mathbf{w}$

### Problem 2.

Solve the vector equation  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$  for the vectors below. Is  $\mathbf{b}$  a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ ?

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 7 \\ -8 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

### Problem 3.

Write the vector equation  $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$  in matrix form, and use this to solve the equation:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 7 \\ a \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

### Problem 4.

Solve the matrix equation  $A\mathbf{x} = \mathbf{b}$  when

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ -1 & 1 & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$$

### Problem 5.

Consider the matrices

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ -1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & 2 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 4 \\ 1 & -2 \\ 7 & 1 \end{pmatrix}$$

Compute the following expressions, whenever possible:

a)  $A + B$       b)  $2A - 3B$       c)  $A - C$       d)  $AB$       e)  $BC$       f)  $ABC$   
 g)  $AC$       h)  $A^2$       i)  $BA$       j)  $CB$       k)  $C^2$       l)  $C^T A$

**Problem 6.**

Determine all  $(a,b,c,d)$  such that the vector  $\mathbf{b}$  is a linear combination of the vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  given below. Use this to determine whether  $\mathbf{b}$  is a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  when  $(a,b,c,d) = (0,0,1,1)$ .

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 3 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 7 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

**Problem 7.**

You have 400.000 NOK and are to invest in a portfolio of risky assets. You can chose a combination of the stocks A, B, C with prices  $p_A = 60$  NOK,  $p_B = 75$  NOK and  $p_C = 320$  NOK per stock at the time of investment. We assume that at a given time in the future, one of three scenarios will happen. The prices of the stocks in these scenarios are given in the table below. Denote by  $x, y, z$  the number of stocks you buy of each of the three risky

	Price A	Price B	Price C
Purchase price	60	75	320
Scenario 1	80	80	350
Scenario 2	100	25	500
Scenario 3	40	100	55

assets. For simplicity, we assume that  $x, y, z$  can be arbitrary real numbers. Hence, we allow for buying a negative number of stocks (short selling), and the number of stocks does not have to be an integer.

- Assume that the condition  $60x + 75y + 320z = 400.000$  is satisfied. What does this condition mean?
- Denote by  $R_1, R_2$  and  $R_3$  the profit from the portfolio in the three scenarios. Is it possible to chose the portfolio such that  $(R_1, R_2, R_3) = (50.000, 25.000, -100.000)$ ? Which portfolio must we chose in order for this to hold?
- Is it possible to chose a portfolio of risky assets such that  $R_1 > 0$  and  $R_2 = R_3 = 0$ ? Which portfolio must we chose in this case? Give an interpretation of your answer.
- Describe all  $(R_1, R_2, R_3)$  of possible profits in the three scenarios. Are there any portfolios such that  $R_1, R_2, R_3 > 0$  (i.e., guaranteed profit in all scenarios)?

**Problem 8.**

Let  $A$  be a  $2 \times 3$ -matrix.

- Is  $A$  symmetric?
- Is  $A^T A$  symmetric?
- Compute  $A^T A$  when  $A = \begin{pmatrix} 1 & -1 & 3 \\ 3 & 3 & 1 \end{pmatrix}$ .

**Problem 9.**

Solve the matrix equation for  $X$  when  $A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$ :

- $AX = I$
- $X^2 = A$
- $AX = XA$

## Optional: Exercises from the Norwegian textbook.

Textbook [E]: Eriksen, *Matematikk for økonomi og finans*

Exercise book [O]: Eriksen, *Matematikk for økonomi og finans - Oppgaver og Løsningsforslag*

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Exercises: [E] 6.5.1 - 6.5.6

Solution manual: See [O] Kap 6.5

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### Answers to exercise session problems

#### Problem 1.

a)  $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$       b)  $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$       c)  $\begin{pmatrix} 4 \\ -4 \end{pmatrix}$       d)  $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$       e)  $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$       f)  $\begin{pmatrix} 2 \\ 11 \end{pmatrix}$

#### Problem 2.

The general solution is  $(x,y,z) = (-4z-1, z+1, z)$  with  $z$  free. A specific solution is found by (for instance) letting  $z = 0$ , which gives  $(-1,1,0)$ . This means that  $\mathbf{b} = -1 \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2$  is a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ .

#### Problem 3.

For  $a = -8$  there are infinitely many solutions  $(x,y,z) = (-4z-1, z+1, z)$  with  $z$  free (like in the previous exercise). For  $a \neq -8$  there is exactly one solution  $(x,y,z) = (-1,1,0)$ .

#### Problem 4.

Exactly one solution  $(x,y,z) = (-3/2, 4, -1/2)$ .

#### Problem 5.

a)  $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 3 \\ 0 & 3 & 5 \end{pmatrix}$       b)  $\begin{pmatrix} 2 & -1 & -4 \\ 1 & 2 & 11 \\ -5 & -4 & -10 \end{pmatrix}$       c) not defined      d)  $\begin{pmatrix} 2 & 3 & 5 \\ 5 & 10 & 19 \\ 2 & 1 & 1 \end{pmatrix}$

e)  $\begin{pmatrix} 15 & 0 \\ -4 & 3 \\ 33 & 4 \end{pmatrix}$       f)  $\begin{pmatrix} 44 & 7 \\ 158 & 19 \\ 14 & 7 \end{pmatrix}$       g)  $\begin{pmatrix} 11 & 3 \\ 35 & 10 \\ 5 & -5 \end{pmatrix}$       h)  $\begin{pmatrix} 2 & 3 & 6 \\ 0 & 7 & 10 \\ 0 & 1 & 4 \end{pmatrix}$

i)  $\begin{pmatrix} 0 & 3 & 6 \\ 2 & 0 & 0 \\ 1 & 7 & 13 \end{pmatrix}$       j) not defined      k) not defined      l)  $\begin{pmatrix} -2 & 11 & 14 \\ -1 & 3 & -3 \end{pmatrix}$

#### Problem 6.

This is a linear combination if  $-7a + 9b - 5c + 3d = 0$ , and  $(a,b,c,d) = (0,0,1,1)$  does not correspond to a linear combination since these values do not satisfy the equation.

#### Problem 7.

- This is a budget constraint, i.e., that the total cost of the risky assets we purchase is 400.000 NOK.
- Yes, if we choose the portfolio  $(x,y,z) = (1187^{1/2}, 2250, 500)$ .
- Yes, if  $R_1 = 80.000$ . Then, we have to choose the portfolio  $(x,y,z) \approx (3333^{1/3}, 2666^{2/3}, 0)$ . This means that we can invest without the risk of loss, and with positive expected profit (a very attractive situation!).

d. The triplets of profit  $(R_1, R_2, R_3)$  that are attainable satisfy the equation  $5R_1 - 2R_2 - 2R_3 = 400.000$ . We can choose the portfolio such that  $R_1, R_2, R_3 > 0$  (i.e., guaranteed profit in all scenarios, an even more attractive situation!). For instance, we can attain the profit  $R_1 = R_2 = R_3 = 400.000$ .

**Problem 8.**

- a) No                                      b) Yes                                      c)  $\begin{pmatrix} 10 & 8 & 6 \\ 8 & 10 & 0 \\ 6 & 0 & 10 \end{pmatrix}$

**Problem 9.**

- a)  $X = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$                                       b) no solution                                      c)  $X = s \begin{pmatrix} -4 & 3 \\ 2 & 0 \end{pmatrix} + t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$