Interpretation of Lagrange multipliers ctd.

EBA 1180
S26
Lect. 48

From last time:

$$\boxed{\text{RESULT:} \quad \lambda = \frac{d f^*(a)}{da}}$$

Ex. ctd: USE THIS TO APPROXIMATE OPTIMAL VALUES

$\underset{\sim}{a=2}$:

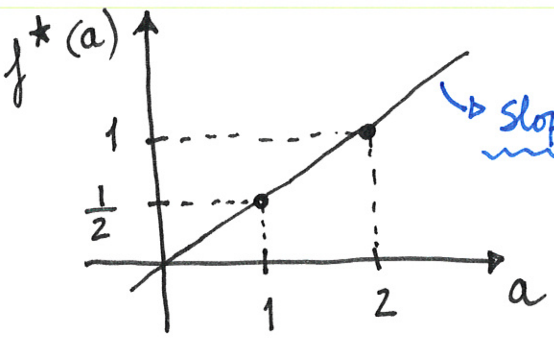
$$f^*(2) \approx f^*(1) + \Delta a \underbrace{\frac{d f^*(a)}{da}}_{= \lambda \text{ from RESULT}}$$

Def. of the derivative

$$= \frac{1}{2} + 1 \cdot \lambda$$

Ex. from last time

$$= \frac{1}{2} + 1 \cdot \frac{1}{2} = \underline{\underline{1}}$$



Slope: $\frac{1}{2} = \lambda = \frac{d f^*(a)}{a}$

Ex ctd.: $\max f(x,y) = xy$ when $\underbrace{x^2 + y^2 = a}_{\text{Circle, center } (0,0),\ r = \sqrt{a},\ a>0}$

EVT? • $f(x,y)$ continuous? Yes.

• Compact constraint set? → Closed? Yes (=)

→ Bounded? Yes (circle)

$\Rightarrow$ EVT holds! The problem has a max (and a min).

Type ii) points: Admissible points with degenerate constraint

$g(x,y) = x^2 + y^2$

$g'_x = 2x = 0 \Rightarrow x = 0$

$g'_y = 2y = 0 \Rightarrow y = 0$

Then, $x^2 + y^2 = 0^2 + 0^2 = 0 \neq a$, $a > 0$

No admissible points with degenerate constraint.

Type i points: $\mathcal{L}(x,y;\lambda) = xy - \lambda(x^2 + y^2 - a)$

FOC:

$$\boxed{\begin{aligned} &(1):\ \mathcal{L}'_x = y - 2x\lambda = 0 \\ &(2):\ \mathcal{L}'_y = x - 2y\lambda = 0 \\ &(3):\ x^2 + y^2 = a \end{aligned}}$$

C: (3)

$\Rightarrow y = 2\lambda x$

$\Rightarrow x - 2 \cdot 2\lambda x \cdot \lambda = 0$

$x - 4\lambda^2 x = 0$

$x \cdot (1 - 4\lambda^2) = 0$

3 CASES

$x = 0$:

From (1):

$y - 2 \cdot \lambda \cdot 0 = 0$

$y = 0$

From (3):

$0^2 + 0^2 = a$

$0 = a$

For $a > 0$: No candidates

(For $a = 0$: $(0,0;\lambda) \to f = xy = 0$)

$\lambda^2 = \frac{1}{4}$:

$\lambda = \frac{1}{2}$:

From (1):

$y = 2x \cdot \frac{1}{2} = x$

From (3): $y^2 + y^2 = a$

$2y^2 = a$

$y^2 = \frac{a}{2}$

$\lambda = -\frac{1}{2}$:

See next page

$\lambda = \frac{1}{2}$ ctd:

$$y = \pm\sqrt{\frac{a}{2}} = x$$

Candidates:

$$\left(\sqrt{\frac{a}{2}}, \sqrt{\frac{a}{2}}; \frac{1}{2}\right) \rightarrow f = \frac{a}{2},$$

$$\left(-\sqrt{\frac{a}{2}}, -\sqrt{\frac{a}{2}}; \frac{1}{2}\right) \rightarrow f = \frac{a}{2}$$

$\lambda = -\frac{1}{2}$ ctd:

From (1): $y = 2x \cdot \left(-\frac{1}{2}\right)$

$$= -x$$

From (3): $y^2 + y^2 = a$

$$2y^2 = a$$

$$y^2 = \frac{a}{2}$$

$$y = \pm\sqrt{\frac{a}{2}}$$

Candidates:

$$\left(\sqrt{\frac{a}{2}}, -\sqrt{\frac{a}{2}}; -\frac{1}{2}\right) \rightarrow f = -\frac{a}{2},$$

$$\left(-\sqrt{\frac{a}{2}}, \sqrt{\frac{a}{2}}; -\frac{1}{2}\right) \rightarrow f = -\frac{a}{2}$$

CONCLUSION:

$f_{max} = \frac{a}{2}$ at the max. points $\left(\sqrt{\frac{a}{2}}, \sqrt{\frac{a}{2}}\right)$ and $\left(-\sqrt{\frac{a}{2}}, -\sqrt{\frac{a}{2}}\right)$ with $\lambda = \frac{1}{2}$.

NOTE: $f^*(a) = f_{max} = \frac{a}{2} = \frac{1}{2}a$

$$\boxed{\frac{d f^*(a)}{da} = \frac{1}{2} = \lambda}$$

RESULT holds for this example (see pg. 1)

Kuhn - Tucker problems

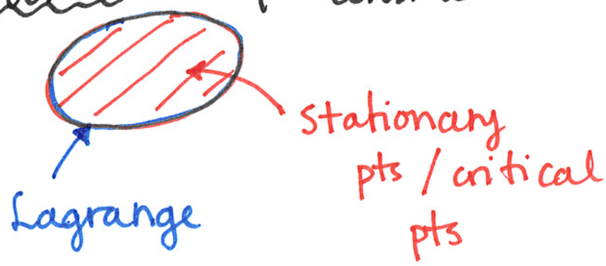
Optimization problems with closed inequality constraints

$(\leq \;,\; \geq)$

Ex:
$$\boxed{\max \quad f(x,y) = x^2 \cdot y^2 \quad \text{when} \quad x^2 + y^2 + x^2 y^2 \leq 3}$$

Kuhn - Tucker problem

Alt. ex: Ellipse constraint



Candidate points

(1) **Boundary points:**

$\Rightarrow$ = constraint

$\Rightarrow$ Lagrange problem

$\Rightarrow$ Solve via std. Lagrange technique.

(2) **Interior points:** Stationary pts. or other critical points. NB: Check if candidates are actually interior pts.

START 11.05

Ex: Stationary pts. or other critical pts:

$$f'_x = 2xy^2 = 0 \quad \Rightarrow \quad x = 0 \text{ or } y = 0$$

(or both)

$$f'_y = 2x^2 y = 0 \quad \Rightarrow \quad x = 0 \text{ or } y = 0$$

NB: No pts. where f'_x or f'_y are not defined

Candidates: i) $x=0$: INTERIOR POINT:

$(0,y)$ when $0^2+y^2+0^2y^2 < 3$

$$y^2 < 3$$

$$-\sqrt{3} < y < \sqrt{3}$$

ii) $y=0$: INTERIOR POINT:

$(x,0)$ when $x^2+0^2+x^2\cdot 0^2 < 3$

$$x^2 < 3$$

$$-\sqrt{3} < x < \sqrt{3}$$

The boundary: Lagrange problem

Boundary

$$\max f(x,y) = x^2y^2 \text{ when } x^2+y^2+x^2y^2 = 3$$

Type i): $\mathcal{L}(x,y:\lambda) = x^2y^2 - \lambda\left(x^2+y^2+x^2y^2-3\right)$

FOC:

$$\mathcal{L}'_x = 2xy^2 - \lambda\,(2x+2xy^2) = 0 \quad (1)$$

$$\mathcal{L}'_y = 2x^2y - \lambda\,(2y+2yx^2) = 0 \quad (2)$$

C:

$$x^2+y^2+x^2y^2 = 3 \quad (3)$$

From (1): $2x\cdot(y^2-\lambda-\lambda y^2)=0 \rightarrow x=0$ OR $\rightarrow y^2-\lambda-\lambda y^2=0$

From (2): $2y\cdot(x^2-\lambda-\lambda x^2)=0 \rightarrow y=0$ OR $\rightarrow x^2-\lambda-\lambda x^2=0$ (or both)

Check all combinations

a) $x=0,\ y=0$: From (3): $0^2+0^2+0^2 0^2=3$

$$0=3$$

Not true $\Rightarrow$ No candidates.

b) $x=0,\ x^2-\lambda-\lambda x^2=0$:

$x=0$:

$$0^2-\lambda-\lambda\cdot 0^2=0$$
$$-\lambda=0$$
$$\lambda=0$$

From (3): $0^2+y^2+0^2y^2=3$

$$y^2=3$$
$$y=\pm\sqrt{3}$$

Candidates: $(0,\sqrt{3};0)$, $(0,-\sqrt{3};0)$

$\boxed{f=x^2y^2=0}$ $\boxed{f=0}$

c) $y=0,\ y^2-\lambda-\lambda y^2=0$: Symmetry (or same again)

Candidates: $(\sqrt{3},0;0)$, $(-\sqrt{3},0;0)$

$\boxed{f=0}$ $\boxed{f=0}$

d) $y^2 - \lambda - \lambda y^2 = 0, \; x^2 - \lambda - \lambda x^2 = 0$:

$$y^2 = \lambda(1+y^2) \Rightarrow \lambda = \frac{y^2}{1+y^2}$$ → never 0

$$x^2 = \lambda(1+x^2) \Rightarrow \lambda = \frac{x^2}{1+x^2}$$ → never 0 (★)

$$\lambda = \lambda$$

$$\frac{y^2}{1+y^2} = \frac{x^2}{1+x^2}$$

$$y^2(1+x^2) = x^2(1+y^2)$$

$$y^2 + \cancel{y^2x^2} = x^2 + \cancel{x^2y^2}$$

$$y^2 = x^2$$

$$y = \pm x \quad (\tilde{★})$$

From (3): $x^2 + x^2 + x^2x^2 = 3$

$$x^4 + 2x^2 - 3 = 0$$

TRICK: Let $u = x^2$

$u^2 + 2u - 3 = 0$; a quadratic eq.

$u = \ldots$ abc-formula/ whatever you prefer $\ldots = \begin{cases} 1 \\ -3 \end{cases}$

So: $x^2 = 1$ or $x^2 = -3$

$x = \pm 1$ Not possible!

Hence, $y \overset{(\tilde{*})}{=} \pm x = \pm 1$

But NOTE: 4 combinations:

$x = 1$: $y = 1$, $y = -1$

$x = -1$: $y = -1$, $y = 1$

Also: $\lambda \overset{(*)}{=} \dfrac{x^2}{1 + x^2} = \dfrac{1}{2}$

Candidates: $(1, 1; \frac{1}{2})$, $(1, -1; \frac{1}{2})$, $(-1, -1; \frac{1}{2})$,

$(-1, 1; \frac{1}{2})$

For all of these: $\boxed{f = x^2 y^2 = 1}$

Type ii) candidates: Admissible pts. with degenerate constraints

$g(x, y) = x^2 + y^2 + x^2 y^2$

$$g'_x = 2x + 2xy^2 = 0 \Rightarrow 2x\,(\overbrace{1+y^2}^{\text{Never }0}) = 0 \Rightarrow x = 0$$

$$g'_y = 2y + 2x^2y = 0 \Rightarrow 2y\,(\underbrace{1+x^2}_{\text{Never }0}) = 0 \Rightarrow y = 0$$

$$\Rightarrow (x,y) = (0,0)$$

Admissible? $g(0,0) = 0^2 + 0^2 + 0^2 0^2 = 0 \neq 3$, so the constraint doesn't hold.

Hence, no candidates of type ii).

Is there a max?

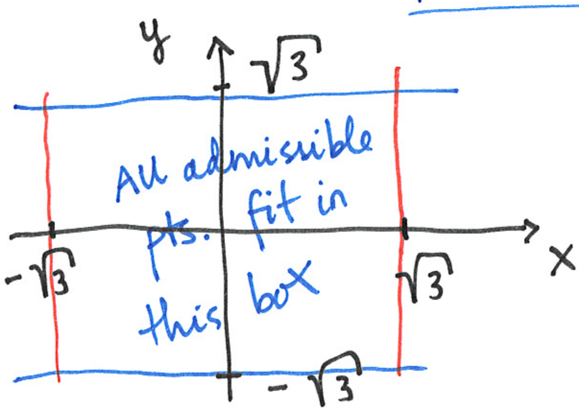
D: $x^2 + y^2 + x^2y^2 \leq 3$

- Closed? Yes
- Bounded? Note that x^2, y^2 and $x^2y^2 \geq 0$ for all x, y.

Hence, both x^2 and y^2 must be ≤ 3.

So: $x^2 \leq 3 \iff \boxed{-\sqrt{3} \leq x \leq \sqrt{3}}$

and $y^2 \leq 3 \iff \boxed{-\sqrt{3} \leq y \leq \sqrt{3}}$



So D is bounded

$\Rightarrow$ D is compact and f is continuous

$\Rightarrow$ EVT holds $\Rightarrow$ There exists a max (and a min).

Conclusion: $f_{max} = 1$ at $(1,1), (1,-1), (-1,1), (-1,-1)$ with $\underline{\underline{\lambda = \frac{1}{2}}}$.