Lagrange problems ctd.

EBA 1180
S26
Lecture 47

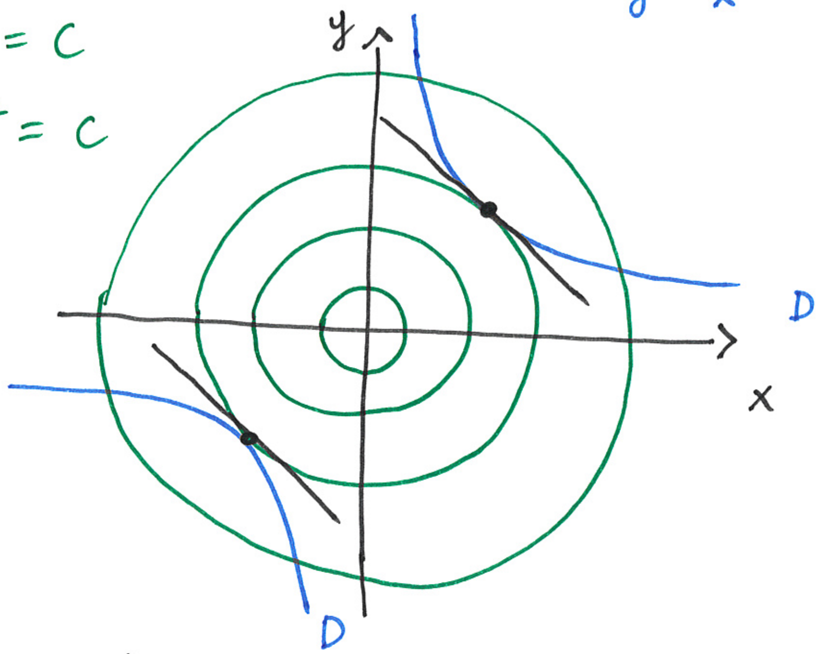
Ex, ctd. from last lecture:

max/min $f(x,y) = x^2 + y^2$ when $\overbrace{xy = 1}^{D}$

$\hookrightarrow y = \frac{1}{x}$

Level curves of f: $f(x,y) = c$

$x^2 + y^2 = c$



From last time:

Can't use EVT since D is not bounded.

Type i) candidates:

$$\mathcal{L}(x,y,\lambda) = x^2 + y^2 - \lambda(xy - 1)$$

FOC:

$$\mathcal{L}'_x = 2x - \lambda y = 0$$

$$\mathcal{L}'_y = 2y - \lambda x = 0$$

C:

$$xy = 1$$

3 eqns., 3 unknowns:

x, y, λ

$$2x = \lambda y$$

$$x = \frac{\lambda y}{2}$$

$$2y - \lambda\left(\frac{\lambda y}{2}\right) = 0 \quad \Big| \cdot 2$$

$$4y - \lambda^2 y = 0$$

$$y \cdot (4 - \lambda^2) = 0$$

3 cases:

$y=0$:

$$x = \frac{\lambda \cdot 0}{2} = 0$$

c: $xy = 0 \cdot 0 = 0 \neq 1$

Not a candidate pt. because constraint doesn't hold.

$\lambda = 2$:

$$x = \frac{2y}{2} = y$$

c: $xy = 1$

$x^2 = 1$

$x = \pm 1$

Candidate pts:

$(1,1)$ and $(-1,-1)$

$f = x^2+y^2 = 1^2+1^2 = 2$

$f = 2$

$\lambda = -2$:

$$x = \frac{-2y}{2} = -y$$

c: $xy = 1$

$-y^2 = 1$

$\underbrace{y^2 = -1}$

NOT POSSIBLE! No candidate pts.

Type ii) candidates: Admissible pts. with degenerate constraint? $\boxed{g(x,y) = xy}$

$$\left.\begin{aligned} g'_x &= y = 0 \\ g'_y &= x = 0 \end{aligned}\right\} \Rightarrow \quad \text{c: } xy = 0 \cdot 0 = 0 \neq 1$$

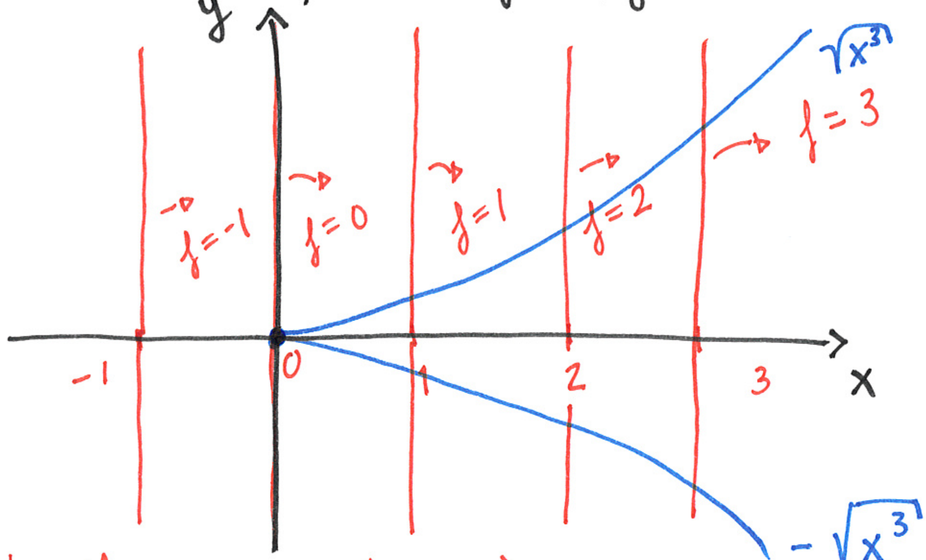
Hence; No admissible points with degenerate constraint.

Conclusion: $f_{min} = 2$ at the pts. $(1,1)$ and $(-1,-1)$ with $\lambda = 2$. This is a min. from figure (smallest level curve/circle while staying on D) or since f is sum of squares, which can't be made infinitely neg.

No maximum from figure or since $y = \frac{1}{x}$ will satisfy the constraint. Let $x \to \infty$. Then, $y \to 0$, but is admissible. Then,

$$f(x,y) = x^2 + y^2 \xrightarrow[x \to \infty]{} \text{“} \infty^2 + 0^2 \text{”} = \infty$$

Ex: max/min $f(x,y) = x$ when $\underbrace{y^2 - x^3 = 0}$



D: $y^2 = x^3$

$y = \pm\sqrt{x^3}$

NB: Only def. for $x^3 \geq 0 \Rightarrow x \geq 0$

START: 11.01

Level curves: $f(x,y) = x = c$

Can't use EVT because D is unbounded.

Type ii) candidates: Admissible pts. with degenerate constraint?

$g(x,y) = y^2 - x^3$

$g'_x = -3x^2 = 0 \Rightarrow x = 0$

$g'_y = 2y = 0 \Rightarrow y = 0$, but then

$\lambda \quad \lambda$

$g(x,y) = g(0,0) = 0^2 - 0^3 = 0$, so $(0,0)$ is on D.

Hence, $(0,0)$ is an admissible pt. with degenerate constraint.

Type i) candidates: $\mathcal{L}(x,y;\lambda) = x - \lambda(y^2 - x^3)$

FOC:
$$\mathcal{L}'_x = 1 + \lambda \cdot 3x^2 = 0 \quad : (1)$$
$$\mathcal{L}'_y = -\lambda \cdot 2y = 0 \quad : (2)$$

C:
$$y^2 - x^3 = 0 \quad : (3)$$

(2): $-\lambda \cdot 2y = 0$

$\lambda = 0$:

(1): $1 + 0 \cdot 3x^2 = 0$

$1 = 0$

Not true! No candidate pt.

$y = 0$:

(3): $0^2 - x^3 = 0 \Rightarrow x = 0$

(1): $1 + \lambda \cdot 3 \cdot 0^2 = 0$

$1 = 0$

Not true! No candidate pt.

Hence, we have no ordinary candidate pts. Only admissible pt. with degenerate constraint; $(0,0)$.

Conclusion: From the figure, we see that this is a minimum pt. (on the smallest level curve possible while on D). No maximum: From figure or no candidates or $y = \pm\sqrt{x^3}$ makes constraint hold. Can choose

D intersects level curves with larger and larger value arbitrarily large $x \Rightarrow f(x,y) = x \xrightarrow[x\to\infty]{} \infty \Rightarrow$ No max.

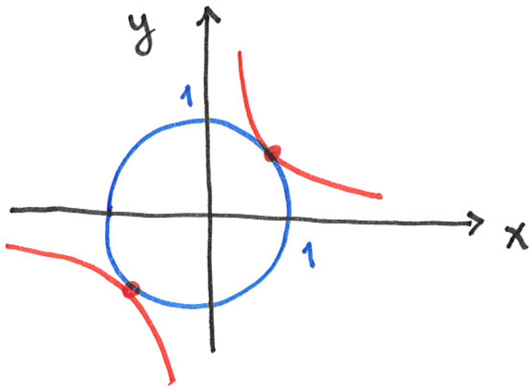
Interpretation of Lagrange multipliers

Ex. from a couple of lect. ago: max/min $f(x,y) = xy$

when $x^2 + y^2 = 1$ → Circle with center $(0,0)$, $r=1$

$f_{max} = \frac{1}{2}$ at

$\left(\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right), \left(-\sqrt{\tfrac{1}{2}}, -\sqrt{\tfrac{1}{2}}\right)$ with $\lambda = \frac{1}{2}$



Parameter: Radius of circle $\sqrt{a}$, $a \geq 0$

Consider: max $f(x,y) = xy$ with $x^2 + y^2 = a$

Max pt: $(x^*(a), y^*(a))$

Read: "f star of a"

Max value: $f(x^*(a), y^*(a)) = f^*(a)$

Ex: $\underset{\sim}{a=1}$: $x^*(1) = \sqrt{\frac{1}{2}}$, $y^*(1) = \sqrt{\frac{1}{2}}$, $f^*(1) = \frac{1}{2}$

OR: $x^*(1) = -\sqrt{\frac{1}{2}}$, $y^*(1) = -\sqrt{\frac{1}{2}}$, $f^*(1) = \frac{1}{2}$

RESULT:

$$\boxed{\lambda = \underbrace{\frac{d f^*(a)}{d a}}_{\text{derivative of optimal value function wrt. parameter } a}}$$

Lagrange multiplier

Roughly: Change in max value when a changes

A small change in a

Interpretation of λ: λ is the marginal change in the max. (min.) value per unit change in the constant a of the constraint $g(x,y) = a$.

Use this to approximate optimal values: Ex. ctd.

$\underset{\sim}{a=2}$: $f^*(2) \approx f^*(1) + \underbrace{\Delta a}_{\text{change in parameter: } 2-1=1} \underbrace{\frac{d f^*(a)}{d a}}_{= \lambda \text{ from RESULT}}$

Def. of derivative:

$f'(a) \approx \frac{f(a+h) - f(a)}{h}$

$f'(a) = \frac{\Delta f(a)}{\Delta a}$

$\Delta a \, f'(a) = \Delta f(a)$

$\Delta f^*(a) \approx \Delta a (f^*)'(a)$

$f^*(2) - f^*(1) \approx \Delta a \frac{d f^*(a)}{d a}$